Computer Science Foundation Exam

August 8, 2008

Section II B

DISCRETE STRUCTURES

KEY

Question #	Category	Max Score	Passing Score	Score
Q4	**CTG (Counting)**	**15**	**10**	
Q5	**PRF (Functions)**	**15**	**10**	
Q6	**PRF (Relations)**	**15**	**10**	
Q7	**NTH (Number Theory)**	**15**	**10**	
ALL	**---**	**60**	**40**	

PART B

3) (CTG) Counting

How many strings of 7 lowercase letters from the English alphabet (of 26 letters) contain

(a) (6 pts) the letter *d* at least once?

(b) (9 pts) the letters *a*, *b*, and *c*, in that order, with all letters distinct? For example, *daebyxc* is a valid string, because all letters are distinct, and *a*, *b*, and *c* appear in order within the string.

Solution.

(a) Count the strings that do not contain the letter *d*. There are 25^7 such strings.
(3 pts)

If we subtract this number from the total number of all possible strings, which is 26^7 (2 pts), we get the number of strings with at least one *d*, namely, $26^7 - 25^7$.
(1 pt)

(b) We can first count all possible ways to place *a*, *b*, and *c* in a 7-letter string. We must choose three slots out of seven for *a*, *b*, and *c* to be placed. There are C(7,3) = 35 ways to do this. Note that we use combinations instead of permutations, because the relative order of *a*, *b*, and *c* is fixed.
(4 pts)

The second step is to choose letters for the remaining four positions without repetition. This task can be done in $23 \cdot 22 \cdot 21 \cdot 20$ different ways.
(4 pts)

Thus, the final answer is $35 \cdot 23 \cdot 22 \cdot 21 \cdot 20$. The two results are multiplied (Product Rule) because each placement of *a*, *b*, and *c* can be combined with a placement of the remaining four letters to create a unique 7-letter string.
(1 pt)

4) (PRF) Relations

Let $S = \{-2, -1, 0, 1\}$ and let $A = S \times S$. Define the following relation R on A:

$$R = \{((a, b), (c, d)) \mid a - b = c - d\}$$

(a) (8 pts) Show that R is an equivalence relation.

(b) (7 pts) Find the partition A/R.

Solution.

(a) (8 pts)

R is reflexive, because for any $(a, b) \in A$ we have $a - b = a - b$, so $((a, b), (a, b)) \in R$. (2 pts)

R is symmetric, because if $((a, b), (c, d)) \in R$, then $a - b = c - d$, by the definition of R. But this means that $c - d = a - b$ as well, i.e., $((c, d), (a, b)) \in R$. (3 pts)

To show that R is transitive let $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$. We need to show that $((a, b), (e, f)) \in R$. By the given definition of R, $((a, b), (c, d)) \in R$ implies that $a - b = c - d$, and $((c, d), (e, f)) \in R$ implies that $c - d = e - f$. From these two equalities we have that $a - b = e - f$, i.e. $((a, b), (e, f)) \in R$. (3 pts)

(b) (7 pts)

$A/R = \{ \{(1, -2)\},$
$\{(1, -1), (0, -2)\},$
$\{(1, 0), (0, -1), (-1, -2)\},$
$\{(1, 1), (0, 0), (-1, -1), (-2, -2)\},$
$\{(0, 1), (-1, 0), (-2, -1)\},$
$\{(-1, 1), (-2, 0)\},$
$\{(-2, 1)\} \}.$

(1 pt for each of the seven equivalence classes)

5) (PRF) Functions

(15 pts) Let $f(x) = x^2 - 5x$, for all real $x \geq 5$. Find $f^{-1}(x)$ and state both the domain and range of $f^{-1}(x)$.

Solution.

Solve for *x* in the given function.

$$f(x) = x^2 - 5x$$

$$f(x) + \frac{25}{4} = x^2 - 5x + \frac{25}{4} \qquad \text{(3 pts)}$$

$$f(x) + \frac{25}{4} = \left(x - \frac{5}{2}\right)^2 \qquad \text{(3 pts)}$$

$$\sqrt{f(x) + \frac{25}{4}} = x - \frac{5}{2} \qquad \text{(3 pts)}$$

$x = \frac{5}{2} + \sqrt{f(x) + \frac{25}{4}}$, thus, $f^{-1}(x) = \frac{5}{2} + \sqrt{x + \frac{25}{4}}$. (2 pts)

Domain of $f^{-1}(x)$ is all real $x \geq 0$. (2 pts)

Range of $f^{-1}(x)$ is all real $x \geq 5$. (2 pts)

6) (NTH) Number Theory

(a) (5 pts) If the product of two integers is $11^2 \cdot 13^2 \cdot 17 \cdot 19^5$ and their least common multiple is $11 \cdot 13 \cdot 17 \cdot 19^3$, what is their greatest common divisor?

(b) (10 pts) Show that gcd(486, 741) can be represented as a linear combination of 486 and 741. (In other words, find integers x and y such that $486x+741y$ = gcd(486, 741).)

Solution.

(a) (5 pts)

If a and b are positive integers, $a \cdot b = \gcd(a, b) \cdot \mathrm{lcm}(a, b)$. (2 pts)
Thus, $\gcd(a, b) = (a \cdot b)/\mathrm{lcm}(a, b) = 11^{(2-1)} \cdot 13^{(2-1)} \cdot 17^{(1-1)} \cdot 19^{(5-3)} = 11 \cdot 13 \cdot 19^2$. (3 pts)

(b) (10 pts)

a	b	r	q
1	0	741	
0	1	486	1
1	-1	255	1
-1	2	231	1
2	-3	24	9
-19	29	15	1
21	-32	9	1
-40	61	6	1
61	-93	3	2

(1 pt for each of the nine rows of the table.)

Therefore $\gcd(486, 741) = 3 = 486(-93) + 741(61)$. (1 pt)