Computer Science Foundation Exam

August 8, 2008

Section II A

DISCRETE STRUCTURES

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PART A

1) (25 pts) PRF (Induction)

Define a sequence of numbers as follows. Let $W_i$ denote the $i^{th}$ W-number. In particular, $W_0 = 2$, $W_1 = 1$, and $W_n = W_{n-1} + \frac{W_{n-2}}{2}$ for all integers $n > 1$.

For all positive integers $n$, prove that $\sum_{i=1}^{n} \frac{W_{i-1}}{W_iW_{i+1}} = 2 - \frac{2}{W_{n+1}}$.

Solution.

**Base case:** $n = 1$. LHS = \[ \sum_{i=1}^{1} \frac{W_{i-1}}{W_iW_{i+1}} = \frac{W_0}{W_1W_2} = \frac{2}{1 \cdot 2} = 1, \text{ RHS} = 2 - \frac{2}{W_2} = 2 - \frac{2}{2} = 1. \]

LHS = RHS and the base case is proven. (4 pts)

**Inductive hypothesis:** Assume for an arbitrary positive integer $n = k$ that \[ \sum_{i=1}^{k} \frac{W_{i-1}}{W_iW_{i+1}} = 2 - \frac{2}{W_{k+1}}. \] (4 pts)

**Inductive Step:** Prove for $n = k + 1$ that \[ \sum_{i=1}^{k+1} \frac{W_{i-1}}{W_iW_{i+1}} = 2 - \frac{2}{W_{k+2}}. \] (4 pts)

\[
\sum_{i=1}^{k+1} \frac{W_{i-1}}{W_iW_{i+1}} = \left( \sum_{i=1}^{k} \frac{W_{i-1}}{W_iW_{i+1}} \right) + \frac{W_k}{W_{k+1}W_{k+2}} (2 \text{ pts})
\]

\[
= 2 - \frac{2}{W_{k+1}} + \frac{W_k}{W_{k+1}W_{k+2}} , \text{ using the inductive hypothesis (3 pts)}
\]

\[
= 2 - \frac{2W_{k+2}}{W_{k+1}W_{k+2}} + \frac{W_k}{W_{k+1}W_{k+2}} (1 \text{ pt})
\]

\[
= 2 - \left( \frac{2W_{k+2} - W_k}{W_{k+1}W_{k+2}} \right) (1 \text{ pt})
\]

\[
= 2 - \frac{2W_{k+1}}{W_{k+1}W_{k+2}} , \text{ since } 2W_{k+2} = 2W_{k+1} + \frac{W_k}{2} = 2W_{k+1} + W_k. \text{ (3 pts)}
\]

\[
= 2 - \frac{2}{W_{k+2}}. \text{ (1 pt)}
\]

The inductive step is complete. We have proven by induction that \[ \sum_{i=1}^{n} \frac{W_{i-1}}{W_iW_{i+1}} = 2 - \frac{2}{W_{n+1}} \]
for all positive integers $n$. (2 pts)
2) (15 pts) PRF (Sets)

Let A, B and C be arbitrary sets taken from the positive integers.

Prove or disprove: If \( A \cap B \cap C = \emptyset \), then \( (A \subseteq \overline{B}) \lor (A \subseteq \overline{C}) \).

Solution.

\( A \cap B \cap C = \emptyset \) is the premise.

\(-\exists x, x \in A \cap B \cap C\) by definition of the empty set.

\( \forall x, -\{x \in A \cap B \cap C\} \) by negation of \( \exists \).

\( \forall x, -\{(x \in A \cap B) \land (x \in C)\} \) by the associative property of intersection.

\( \forall x, -\{(x \in A) \land (x \in B)\} \lor -(x \in C) \) by DeMorgan’s Laws.

\( \forall x, -(x \in A) \lor -(x \in B) \lor -(x \in C) \) by the definition of intersection.

\( \forall x, -(x \in A) \lor -(x \in B) \lor -(x \in C) \) by the associative property of intersection.

\( \forall x, (x \not\in A) \lor (x \not\in B) \lor (x \not\in C) \) by negation of \( \in \).

(6 pts)

For some arbitrary element \( y \), assume \( y \in A \).

\(-\neg(y \in A)\) by double negation.

\((y \not\in A) \lor (y \not\in B) \lor (y \not\in C)\) by application of \( \lor \) in \( \forall x, (x \not\in A) \lor (x \not\in B) \lor (x \not\in C) \).

\((y \not\in A) \lor [(y \not\in B) \lor (y \not\in C)]\) by the associative property of \( \lor \).

\((y \not\in B) \lor (y \not\in C)\) by disjunctive syllogism.

\((y \in \overline{B}) \lor (y \in \overline{C})\) by definition of set complement.

(4 pts)

Case 1: \( (y \in \overline{B}) \).

\((y \in A) \rightarrow (y \in \overline{B})\).

\( A \subseteq \overline{B} \) by definition of subset.

(2 pts)

Case 2: \( (y \in \overline{C}) \).

\((y \in A) \rightarrow (y \in \overline{C})\).

\( A \subseteq \overline{C} \) by definition of subset.

(2 pts)

\((A \subseteq \overline{B}) \lor (A \subseteq \overline{C})\) by combining the two cases.

(1 pt)