

# Computer Science Foundation Exam

August 10, 2007

Section II A

## DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,  
and you must work entirely on your own.**

**Name:** \_\_\_\_\_

**SSN:** \_\_\_\_\_

**In this section of the exam, there are three (3) problems.**

**You must do all of them.**

**They count for 40% of the Discrete Structures exam grade.**

**Show the steps of your work carefully.**

**Problems will be graded based on the completeness of the solution steps and  
not graded based on the answer alone.**

**Credit cannot be given when your results are unreadable.**

| <b>Question #</b> | <b>Category</b>           | <b>Score</b> | <b>Max Score</b> |
|-------------------|---------------------------|--------------|------------------|
| <b>1</b>          | <b>PRF (Induction)</b>    |              | <b>15</b>        |
| <b>2</b>          | <b>PRF (Direct Proof)</b> |              | <b>15</b>        |
| <b>3</b>          | <b>PRF (Logic)</b>        |              | <b>10</b>        |
| <b>ALL</b>        | <b>---</b>                |              | <b>40</b>        |

## FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer *all* of Part A and *all* of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

### PART A: Work all of the following problems (1, 2, 3 and 4).

#### 1) (15 pts) (PRF) Induction

The sequence of Fibonacci numbers is defined as follows:  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for all integers  $n$  greater than or equal to 2. The sequence of Lucas numbers is defined as follows:  $L_0 = 2$ ,  $L_1 = 1$ , and  $L_n = L_{n-1} + L_{n-2}$  for all integers  $n$  greater than or equal to 2. Using strong induction on  $n$ , prove that

$$5F_n = L_{n-1} + L_{n+1}$$

for all positive integers  $n$ .

#### Solution.

**Base case:**  $n = 1$ . LHS =  $5F_1 = 5 \cdot 1 = 5$ , RHS =  $L_0 + L_2 = 2 + 3 = 5$ .

Thus the given statement is true for  $n = 1$ . **(1 pt)**

**Base case:**  $n = 2$ . LHS =  $5F_2 = 5 \cdot 1 = 5$ , RHS =  $L_1 + L_3 = 1 + 4 = 5$ .

Thus the given statement is true for  $n = 2$ . **(1 pt)**

**Strong inductive hypothesis:** Assume for  $1 \leq n \leq k$ , where  $k \geq 2$  is an arbitrary positive integer, that

$$5F_n = L_{n-1} + L_{n+1} \cdot \mathbf{(2 pts)}$$

**Inductive step:** We will prove for  $n = k + 1$  that  $5F_{k+1} = L_k + L_{k+2}$ . **(2 pts)**

$$\begin{aligned} 5F_{k+1} &= 5(F_k + F_{k-1}) && \text{by definition of Fibonacci numbers (2 pts)} \\ &= 5F_k + 5F_{k-1} && \text{by the distributive law (1 pt)} \\ &= (L_{k-1} + L_{k+1}) + (L_{k-2} + L_k) && \text{using the IH (2 pts)} \\ &= (L_{k-2} + L_{k-1}) + (L_k + L_{k+1}) && \text{by the associative law (1 pt)} \\ &= L_k + L_{k+2} && \text{by definition of Lucas numbers (2 pts)} \end{aligned}$$

This completes the proof of the inductive step.

**Conclusion:** Therefore,  $5F_n = L_{n-1} + L_{n+1}$  is true for all positive integers  $n$ . **(1 pts)**

2) (15 pts) (PRF) Sets

Let A, B and C be arbitrary sets taken from the universe of integers. Prove the following assertion using direct proof.

Note: You may **not** use the “truth table” or “membership” method for the proof.

$$[A \cap (B - C)] \cup [A \cap (C - B)] = [A \cap (B \cup C)] - (B \cap C)$$

**Solution.**

$$\begin{aligned} & [A \cap (B - C)] \cup [A \cap (C - B)] \\ &= A \cap [(B - C) \cup (C - B)] \quad \text{by the distributive law (2 pts)} \\ &= A \cap [(B \cap \bar{C}) \cup (C \cap \bar{B})] \quad \text{by definition of set difference (1 pt)} \\ &= A \cap [(B \cap \bar{C}) \cup C] \cap [(B \cap \bar{C}) \cup \bar{B}] \quad \text{by the distributive law (2 pts)} \\ &= A \cap [(B \cup C) \cap (\bar{C} \cup C)] \cap [(B \cup \bar{B}) \cap (\bar{C} \cup \bar{B})] \quad \text{by the distributive law (2 pts)} \\ &= A \cap [(B \cup C) \cap U] \cap [U \cap (\bar{C} \cup \bar{B})] \quad \text{by definition of } U \text{ (the universe) (1 pt)} \\ &= A \cap [(B \cup C) \cap (\bar{C} \cup \bar{B})] \quad \text{because } U \cap S = S \text{ for all sets } S \text{ (1 pt)} \\ &= [A \cap (B \cup C)] \cap (\bar{C} \cup \bar{B}) \quad \text{by the associative law (1 pt)} \\ &= [A \cap (B \cup C)] - \overline{(\bar{C} \cup \bar{B})} \quad \text{by the definition of set difference (and double negation) (2 pts)} \\ &= [A \cap (B \cup C)] - (C \cap B) \quad \text{by DeMorgan's Laws (2 pts)} \\ &= [A \cap (B \cup C)] - (B \cap C) \quad \text{by the commutative law. (1 pt)} \end{aligned}$$

**3) (10 pts) (PRF) Logic**

Use a truth table to prove that  $p \rightarrow (q \vee r)$  is logically equivalent to  $(p \rightarrow q) \vee (p \rightarrow r)$ .

**Solution.**

| $p$ | $q$ | $r$ | $q \vee r$ | $p \rightarrow (q \vee r)$ | $p \rightarrow q$ | $p \rightarrow r$ | $(p \rightarrow q) \vee (p \rightarrow r)$ |
|-----|-----|-----|------------|----------------------------|-------------------|-------------------|--|
| 0   | 0   | 0   | 0          | 1                          | 1                 | 1                 | 1  |
| 0   | 0   | 1   | 1          | 1                          | 1                 | 1                 | 1  |
| 0   | 1   | 0   | 1          | 1                          | 1                 | 1                 | 1  |
| 0   | 1   | 1   | 1          | 1                          | 1                 | 1                 | 1  |
| 1   | 0   | 0   | 0          | 0                          | 0                 | 0                 | 0  |
| 1   | 0   | 1   | 1          | 1                          | 0                 | 1                 | 1  |
| 1   | 1   | 0   | 1          | 1                          | 1                 | 0                 | 1  |
| 1   | 1   | 1   | 1          | 1                          | 1                 | 1                 | 1  |

**(1 pt for each row)**

Because the columns for  $p \rightarrow (q \vee r)$  and  $(p \rightarrow q) \vee (p \rightarrow r)$  are identical, we conclude that  $p \rightarrow (q \vee r)$  is logically equivalent to  $(p \rightarrow q) \vee (p \rightarrow r)$ . **(2 pts)**

# Computer Science Foundation Exam

August 10, 2007

Section II B

## DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,  
and you must work entirely on your own.**

**Name:** \_\_\_\_\_

**SSN:** \_\_\_\_\_

**In this section of the exam, there are four (4) problems.  
You must do ALL of them.  
Each counts for 15% of the Discrete Structures exam grade.  
Show the steps of your work carefully.**

**Problems will be graded based on the completeness of the solution steps and  
not graded based on the answer alone.**

**Credit cannot be given when your results are unreadable.**

| Question # | Category            | Score | Max Score |
|------------|---------------------|-------|-----------|
| 4          | CTG (Counting)      |       | 15        |
| 5          | PRF (Direct)        |       | 15        |
| 6          | PRF (Direct)        |       | 15        |
| 7          | NTH (Number Theory) |       | 15        |
| ALL        | ---                 |       | 60        |

**PART B: Work ALL 4 of the problems 4 – 7.**

**4) (CTG) Counting**

(a) (5 pts) How many different three-letter strings can be formed from the letters in the word DOLPHINS? (Note: The order of the letters matters, thus, POD is different from ODP. Also, the number of times a letter appears in the string can NOT exceed the number of times it appears in DOLPHINS.)

(b) (5 pts) A donut shop sells seven different types of donuts, with an unlimited number of each type. How many ways are there to purchase three donuts?

(c) (5 pts) How many distinct binary strings contain five zeroes and three ones?

**Solution.**

(a)

All the letters in DOLPHINS are distinct. **(2 pts)**

Thus the number of three-letter arrangements of the letters in DOLPHINS is  $P(8,3)$ , **(2 pts)**

which equals  $\frac{8!}{(8-3)!} = 8 \cdot 7 \cdot 6 = 336$ . **(1 pt)**

(b)

This problem involves combinations with repetitions, or “balls into bins”, because the donuts represent indistinct balls and the types of donuts represent distinct bins. **(2 pts)**

The number of ways to choose three donuts from seven types is  $C(3 + (7 - 1), (7 - 1)) = C(3 + (7 - 1), 3) = C(9, 6)$ , **(2 pts)**

which equals  $\frac{9!}{6!(9-6)!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$ . **(1 pt)**

(c)

This problem involves combinations with repetitions, or “balls into bins”, because the zeroes represent indistinct balls and the ones represent dividers between distinct bins. **(2 pts)**

The number of ways to place 5 zeroes given a total of  $5 + 3 = 8$  distinct positions is  $C(8, 5)$ , **(2 pts)**

which equals  $\frac{8!}{5!(8-5)!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$ . **(1 pt)**

Another way to solve this problem is first to arrange the zeroes and ones as if they were all distinct ( $8!$ ), then divide by the number of the arrangements of the zeroes ( $5!$ ) multiplied by the number of arrangements of the ones ( $3!$ ).

## 5) (PRF) Relations

Let  $A = \{1,2,3\}$  and let  $S = A \times A$ . Define the following relation  $R$  on  $S$ :

$$R = \{((a,b),(c,d)) \mid a \equiv_3 c \text{ and } b \equiv_2 d\}.$$

Note that  $x \equiv_m y$  if  $x$  and  $y$  are integers and  $m \mid (x - y)$ .

(a) (9 pts) Prove that  $R$  is an equivalence relation.

(b) (6 pts) Give the partition  $(A \times A) / R$ .

### Solution.

(a)

We will prove that  $R$  is reflexive, symmetric, and transitive.

Given arbitrary integers  $a, b \in A$ , notice that  $a \equiv_3 a$  and  $b \equiv_2 b$ , because  $a - a = 0 = 3 \cdot 0$  and  $b - b = 0 = 2 \cdot 0$ . Thus  $((a,b),(a,b)) \in R$  for all integers  $a, b$  and  $R$  is reflexive. **(3 pts)**

Assume for arbitrary integers  $a, b, c, d \in A$  that  $((a,b),(c,d)) \in R$ . By definition of  $R$ , we have  $a \equiv_3 c$  and  $b \equiv_2 d$ . By definition of  $\equiv_m$ , we have  $a - c = 3x$  and  $b - d = 2y$ , where  $x$  and  $y$  are integers. It follows that  $c - a = 3(-x)$  and  $d - b = 2(-y)$ . Thus  $c \equiv_3 a$  and  $d \equiv_2 b$ , and it follows that  $((c,d),(a,b)) \in R$ . Therefore  $R$  is symmetric. **(3 pts)**

Assume for arbitrary integers  $a, b, c, d, e, f \in A$  that  $((a,b),(c,d)) \in R$  and  $((c,d),(e,f)) \in R$ . By definition of  $R$ , we have  $a \equiv_3 c$ ,  $b \equiv_2 d$ ,  $c \equiv_3 e$ , and  $d \equiv_2 f$ . By definition of  $\equiv_m$ , we have  $a - c = 3w$ ,  $b - d = 2x$ ,  $c - e = 3y$ , and  $d - f = 2z$ , where  $w, x, y$ , and  $z$  are integers. By adding equations, it follows that  $a - e = (a - c) + (c - e) = 3w + 3y = 3(w + y)$  and  $b - f = (b - d) + (d - f) = 2x + 2z = 2(x + z)$ . Note that  $w + y$  and  $x + z$  are integers by the closure of the integers under addition. Thus  $a \equiv_3 e$  and  $b \equiv_2 f$ , and it follows that  $((a,b),(e,f)) \in R$ . Therefore  $R$  is transitive. **(3 pts)**

Because  $R$  is reflexive, symmetric, and transitive, we conclude that  $R$  is an equivalence relation.

(b)

$$(A \times A) / R = \{ \{(1,1), (1,3)\}, \{(2,1), (2,3)\}, \{(3,1), (3,3)\}, \{(1,2)\}, \{(2,2)\}, \{(3,2)\} \}$$

**(1 pt for each equivalence class)**

6) (15 pts) (PRF) Functions

Let  $f(x) = \frac{x-3}{2x}$ , with a domain of  $x \in (0, \infty)$ . Determine  $f^{-1}(x)$  and the domain and range of  $f^{-1}(x)$ .

**Solution.**

$$x = \frac{f^{-1}(x) - 3}{2f^{-1}(x)} = \frac{1}{2} - \frac{3}{2f^{-1}(x)} \quad (4 \text{ pts})$$

$$\frac{3}{2f^{-1}(x)} = \frac{1}{2} - x = \frac{1-2x}{2} \quad (4 \text{ pts})$$

$$\frac{3}{f^{-1}(x)} = 1 - 2x \quad (3 \text{ pts})$$

$$f^{-1}(x) = \frac{3}{1-2x} \quad (2 \text{ pts})$$

The range of  $f^{-1}(x)$  is  $f^{-1}(x) \in (0, \infty)$ , because the range of an inverse function is the domain of the original function. (1 pt)

The input values of  $x$  that produce these output values are all values  $x < \frac{1}{2}$ . As  $x$  approaches negative infinity,  $f^{-1}(x)$  approaches 0 from the right. As  $x$  approaches  $\frac{1}{2}$  from the left,  $f^{-1}(x)$  approaches positive infinity. Thus, the domain of this inverse function is  $x \in (-\infty, \frac{1}{2})$  (1 pt)

7) (NTH) Number Theory

(a) (5 pts) Use the Euclidean Algorithm to find  $\gcd(720, 408)$ .

(b) (5 pts) Use the Extended Euclidean Algorithm to express  $\gcd(720, 408)$  as a linear combination of 720 and 408.

(c) (5 pts) Prove that  $\sum_1^n i$  is even if  $4 \mid n$ , where  $n$  is an arbitrary positive integer.

**Solution.**

(a)

$$720 = 408 \cdot 1 + 312$$

$$408 = 312 \cdot 1 + 96$$

$$312 = 96 \cdot 3 + 24$$

$$96 = 24 \cdot 4 + 0$$

so  $\gcd(720, 408) = 24$ . **(1 pt for each step, including the final answer)**

(b)

$$24 = 312 - 96 \cdot 3$$

$$= 312 - (408 - 312) \cdot 3$$

$$= 312 \cdot 4 - 408 \cdot 3$$

$$= (720 - 408) \cdot 4 - 408 \cdot 3$$

$$= 720 \cdot 4 - 408 \cdot 7.$$

**(1 pt for each step)**

(c)

We are given that  $4 \mid n$ , thus we can write  $n = 4a$  for some integer  $a$  by definition of divisibility.

**(1 pt)**

Note that

$$\sum_1^n i = \frac{n(n+1)}{2}. \quad \text{(1 pt)}$$

It follows that

$$\begin{aligned} \sum_1^n i &= \frac{n(n+1)}{2} \\ &= \frac{4a(4a+1)}{2} \\ &= 2a(4a+1) \\ &= 2[a(4a+1)] \end{aligned}$$

**(2 pts)**

Now  $a(4a+1)$  is an integer by the closure of the integers under multiplication and addition.

Therefore  $\sum_1^n i$  is even by definition of an even number. **(1 pt)**