Computer Science Foundation Exam

August 11, 2006

Section II B

DISCRETE STRUCTURES

KEY

PART B: Work any two of the following problems (3 through 6).

3) (CTG) Counting

(a) Consider a set of five distinct computer science books, three distinct math books, and two distinct art books.

(i) (3 pts) In how many ways can these books be arranged on a shelf?
(ii) (7 pts) In how many ways can these books be arranged on a shelf if the two art books are not together?

(b) How many strings of six lowercase letters from the English alphabet contain

(i) (6 pts) the letter *a* at least once?
(ii) (9 pts) the letters *a* and *b*, where *a* is somewhere to the left of *b* in the string, with all letters distinct?

(a) (i) 10!, since each object is distinct. (3 pts all or nothing)

(ii) It can be solved in different ways.

<u>One possible solution</u>. Count all arrangements when 2 art books are together. This is 2 ways to arrange art books (1 pt) between themselves times 9! Permutations (2 pts) of the rest 8 distinct books and the two art books that are kept together, so $9!\cdot 2!$. (1 pt) Then the number of arrangements with two art books not together will be $10! - 9!\cdot 2! = 9!\cdot(10-2) = 9!\cdot 8$ (3 pts)

<u>Another possible solution</u>. Let's arrange 8 books except art books first, we have 8! ways to perform this task. (2 pts) After that we have 9 slots to insert one of the art books between eight other books. (2 pts) The last step is to insert the second art book If we don't want to insert two art books in the same slot, we have 8 ways to perform this task. (2 pts) Finally by the product rule $8!\cdot 9\cdot 8 = 9!\cdot 8$. (1 pt)

(b) (i) Let's count the strings that do not contain letter *a*. There are 25^6 such strings. (3 pts) If we subtract this number from the total number of all possible strings 26^6 (2 pts), we get the number of strings with at least one letter *a*. $26^6 - 25^6$. (1 pt)

(ii) We can count first all possible ways to arrange *a* and *b* in a 6-letter word. We must choose two slots out of six for *a* and *b* to be arranged. We choose instead of permute because the relative order of *a* and *b* are fixed. There are $\binom{6}{2} = 15$, ways to do this. (4 pts) The second step is to choose letters for the remaining 4 positions without repetition. This task can be done in $24!/20!$ different ways. (4 pts) Thus, the final answer is $15\cdot 24!/20!$. The two parts are multiplied because each possible placement of a and b can be combined with placements of the rest of the letters. (1 pt)

4) (PRF) Relations

(a) Let $S = \{1, 2, 3, 4\}$ and let $A = S\times S$. Define the following relation R on A:

$$R = \{((a, b), (c, d)) \mid a + b = c + d\}$$

(i) (7 pts) Show that R is an equivalence relation.
(ii) (6 pts) Find the partition A/R

(b) Suppose R and S are symmetric binary relations on a set A. Must the following relations be symmetric? Give either proofs or counterexamples to justify your answers.

(i) (6 pts) $R \cup S$
(ii) (6 pts) $R\circ S$

(a) (i) R is reflexive, because for any $(a, b) \in A$ we have that $a + b = a + b$, so $((a, b), (a, b)) \in R$. (2 pts)

R is symmetric, because if $((a, b), (c, d))\in R$, then $a + b = c + d$, by the definition of R. But this means that $c + d = a + b$ as well, i.e. $((c, d), (a, b))\in R$. (2 pts)

To show that R is transitive let $((a, b), (c, d))\in R$ and $((c, d), (e, f))\in R$. We need to show that $((a, b), (e, f))\in R$. By the given definition of R, $((a, b), (c, d))\in R$ implies that $a + b = c + d$, and $((c, d), (e, f))\in R$ implies that $c + d = e + f$. From these two equalities we have that $a + b = e + f$, i.e. $((a, b), (e, f))\in R$. (3 pts)

(ii) $A/R = \{\{(1, 1)\}, \{(1, 2), (2, 1)\}, \{(1, 3), (2, 2), (3, 1)\}, \{(1, 4), (2, 3), (3, 2), (4, 1)\}, \{(2, 4), (3, 3), (4, 2)\}, \{(3, 4), (4, 3)\}, \{(4, 4)\}\}$, 2 pt for listing sets of ordered pairs, about 1/2 point for each equivalence class, only award a whole number of points.

(b) (i) If R and S are both symmetric relations, then $R \cup S$ is also symmetric.
To prove this proposition, assume that $(x, y)\in R \cup S$, i.e. (x, y) is an arbitrary element from the set $R \cup S$. We want to prove that $(y, x)\in R \cup S$. (1 pt) From assumption $(x, y)\in R \cup S$ we can imply that either $(x, y)\in R$ or $(x, y)\in S$, by the definition of $\cup$. (2 pts) So, there are two cases to consider. If $(x, y)\in R$, then $(y, x)\in R$ by the symmetric property of R. Then $(y, x)\in R \cup S$ because $R \subseteq R \cup S$. (1 pt) In the other case, if $(x, y)\in S$, then $(y, x)\in S$ by the symmetric property of S. Then $(y, x)\in R \cup S$ since $S \subseteq R \cup S$. (1 pt)Thus, whenever $(x, y)\in R \cup S$, then $(y, x)\in R \cup S$, i.e. the relation $R \cup S$ is symmetric. (1 pt)

(ii) The proposition is false, because it does not hold for all symmetric relations. The following counterexample disproves the proposition that if R and S are symmetric, then $R\circ S$ is symmetric. Let $R = \{(2, 3), (3, 2)\}$, $S =\{(1, 2), (2, 1)\}$ are two symmetric relations on a set $A =\{1, 2, 3\}$. Then the composite relation $R\circ S = \{(1, 3)\}$ is not symmetric. (3 pts false, 3 pts for counter-example.)

5) (PRF) Functions

(a) (15 pts) Let $f(x) = x^2 + 4x$, for all real $x \le -2$. Find $f^{-1}(x)$ and state both the domain and range of $f^{-1}(x)$.

(b) (10 pts) Define a function f(n) for the domain and codomain of all positive integers as follows:

f(n) = the sum of the decimal digits of n

Is f(n) injective? Is f(n) surjective? Give proof for both of your answers.

(a) Solve for x in the given function.

$f(x) = x^2 + 4x$
$f(x) + 4 = x^2 + 4x + 4$ (3 pts)
$f(x) + 4 = (x+2)^2$ (2 pts)
$-\sqrt{f(x)+4} = (x+2)$ (2 pts for sqrt, 2 pts for neg sign)
$x = -2 - \sqrt{f(x)+4}$, thus, $f^{-1}(x) = -2 - \sqrt{x+4}$ (2 pts)
Domain: all real x ≥ -4 (2 pts), Range: all real x ≤ -2. (2 pts)

(b) f is NOT injective because f(25) = f(52) = 7. (5 pts – 1 for answer, 4 for counterexample)

f IS surjective. We must show that for every positive integer n, there exists some positive integer m such that f(m) = n. For any given value of n, simply let m be the decimal number with n consecutive ones in it. For example, for n=5, let m=11111. It is clear that this construction can be carried out for any positive integer n. Thus, we have some that for all possible values in the codomain, there exists a value in the domain that f maps to it. (1 pt for answer, 4 points for showing how all possible values can be mapped.)

6) (NTH) Number Theory

(a) (5 pts) If the product of two integers is $2^7 3^8 5^2 7^{11}$ and their greatest common divisor is $2^3 3^4 5$, what is their least common multiple?

(b) (6 pts) Let a and b be two positive integer. Show that $3ab + 7ab^2$ is even.

(c) (6 pts) Use the Euclidean algorithm to find gcd(56, 147)

(d) (8 pts) Using the Extended Euclidean algorithm, show that the gcd(56, 147) can be represented as a linear combination of 56 and 147. (Namely, determine integers x and y such that 56x+147y = gcd(56, 147).

(a) If $a$and b are positive integers, $a \cdot b$= gcd(a, b)·lcm(a, b). (2 pts) So,
lcm(a, b) = $(a \cdot b)/\gcd(a, b) = 2^{(7-3)} \cdot 3^{(8-4)} \cdot 5^{(2-1)} \cdot 7^{11} = 2^4 \cdot 3^4 \cdot 5 \cdot 7^{11}$ (3 pts)

(b) $3ab + 7ab^2 = ab(3 + 7b)$

If either a or b is even, then $ab(3 + 7b)$ is even. (2 pts) Consider the case when both a and b are odd. In this case, $7b$ would be an odd number (2 pts), making $3+7b$ an even number. (1 pt) Thus, in all possible cases, at least one of the three factors a, b, $(3+7b)$ is even, making $3ab + 7ab^2$ even for all positive integers a and b. (1 pt)

(c) $147=56 \cdot 2+35$ Grading: 1 pt for each line,
$56=35 \cdot 1+21$
$35=21 \cdot 1+14$
$21=14 \cdot 1+7$
$14=7 \cdot 2$

gcd(147, 56)=7 (1 pt for final answer)

(d) $7= 21 - 14$ (1 pt)
$= 21 - (35-21) = 2 \cdot 21-35$ (2 pts)
$=2 \cdot (56 -35) -35 = 2 \cdot 56 -3 \cdot 35$ (2 pts)
$=2 \cdot 56 -3 \cdot (147-56 \cdot 2) = 8 \cdot 56-3 \cdot 147$ (3 pts)