

Computer Science Foundation Exam

August 11, 2006

Section II A

DISCRETE STRUCTURES

KEY

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: Work both of the following problems (1 and 2).

1) (25 pts) (PRF) Induction

Let F_i denote the i^{th} Fibonacci number. In particular, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 1$.

For all positive integers n , prove $\sum_{i=1}^n \frac{F_{i-1}}{F_i F_{i+1}} = 1 - \frac{1}{F_{n+1}}$.

2) (PRF) Sets

Let A , B and C be arbitrary sets taken from the set of positive integers.

(a) (10 pts) Show that the sets $\overline{A \cup B \cap C}$ and $\overline{A} \cap B \cap C$ are equal, using any method.

(b) (15 pts) Prove or disprove: If $C \subseteq B$, then $(A - B) \cup (B - C) \subseteq \overline{C} \cap (A \cup B)$.

Solution to Problem 1:

(PRF) Let F_i denote the i^{th} Fibonacci number. In particular, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 1$.

For all positive integers n , prove $\sum_{i=1}^n \frac{F_{i-1}}{F_i F_{i+1}} = 1 - \frac{1}{F_{n+1}}$.

Base case: $n=1$. LHS = $\sum_{i=1}^1 \frac{F_{i-1}}{F_i F_{i+1}} = \frac{F_0}{F_1 F_2} = \frac{0}{1 \times 1} = 0$, RHS = $1 - \frac{1}{F_2} = 1 - \frac{1}{1} = 0$ (2 pts)

Inductive hypothesis: Assume for an arbitrary positive integer $n=k$ that $\sum_{i=1}^k \frac{F_{i-1}}{F_i F_{i+1}} = 1 - \frac{1}{F_{k+1}}$. (2 pts)

Inductive Step: Prove for $n=k+1$ that $\sum_{i=1}^{k+1} \frac{F_{i-1}}{F_i F_{i+1}} = 1 - \frac{1}{F_{k+2}}$ (2 pts)

$$\sum_{i=1}^{k+1} \frac{F_{i-1}}{F_i F_{i+1}} = \left(\sum_{i=1}^k \frac{F_{i-1}}{F_i F_{i+1}} \right) + \frac{F_k}{F_{k+1} F_{k+2}} \quad (3 \text{ pts})$$

$$= 1 - \frac{1}{F_{k+1}} + \frac{F_k}{F_{k+1} F_{k+2}}, \text{ using the inductive hypothesis (3 pts)}$$

$$= 1 - \frac{F_{k+2}}{F_{k+1} F_{k+2}} + \frac{F_k}{F_{k+1} F_{k+2}} \quad (3 \text{ pts})$$

$$= 1 - \left(\frac{F_{k+2} - F_k}{F_{k+1} F_{k+2}} \right) \quad (3 \text{ pts})$$

$$= 1 - \frac{F_{k+1}}{F_{k+1} F_{k+2}}, \text{ since } F_{k+2} = F_{k+1} + F_k. \quad (4 \text{ pts})$$

$$= 1 - \frac{1}{F_{k+2}} \quad (3 \text{ pts})$$

Solution to Problem 2:

(PRF) Let A, B and C be arbitrary sets taken from the set of positive integers.

(a) (10 pts) Show that the sets $\overline{A \cup B \cap C}$ and $\bar{A} \cap B \cap C$ are equal, using any method.

(b) (15 pts) Prove or disprove: If $C \subseteq B$, then $(A - B) \cup (B - C) \subseteq \bar{C} \cap (A \cup B)$.

(a) In this solutions, set laws will be used to show the equivalence:

$$\begin{aligned}\overline{A \cup B \cap C} &= \bar{A} \cap \overline{B \cap C}, \text{ using DeMorgan's Law (4 pts - 3 for the step, 1 for reason)} \\ &= \bar{A} \cap (B \cap C), \text{ using Law of Double Complement (4 pts - same as above)} \\ &= \bar{A} \cap B \cap C, \text{ using Associative Law (2 pts - 1 pt for the step, 1 reason)}\end{aligned}$$

Another solution uses a Membership Table. For this solution award 1 point for each row of the table, 1 point for setting up the table and 1 point for arriving at the desired conclusion based upon the table.

(b) We must prove that if an arbitrary element $x \in (A - B) \cup (B - C)$, then $x \in \bar{C} \cap (A \cup B)$. We split our work into two cases: (1) $x \in (A - B)$, and (2) $x \in (B - C)$.

Case (1): $x \in (A - B)$, by definition of set difference, we have that $x \in A$, and $x \notin B$. Since the latter is true, and since $C \subseteq B$, it follows that $x \notin C$. Hence, $x \in \bar{C}$, based on the definition of set complement. Also, since $x \in A$, it naturally follows that $x \in (A \cup B)$, by definition of set union. Since $x \in \bar{C}$ and $x \in (A \cup B)$, by definition of set intersection, we conclude that $x \in \bar{C} \cap (A \cup B)$.

Case (2): $x \in (B - C)$, by definition of set difference, we have that $x \in B$, and $x \notin C$. Hence, $x \in \bar{C}$, based on the definition of set complement. Also, since $x \in B$, it naturally follows that $x \in (A \cup B)$, by definition of set union. Since $x \in \bar{C}$ and $x \in (A \cup B)$, by definition of set intersection, we conclude that $x \in \bar{C} \cap (A \cup B)$.

Grading: Stating equivalent if-then statement (2 pts)

Splitting work into two cases (2 pts)

Case (1) – 6 pts (1 for using set diff, 3 for determining $x \in \bar{C}$,
1 for $x \in (A \cup B)$, 1 for conclusion)

Case (2) – 5 pts (Same as above, but 2 pts for $x \in \bar{C}$)

Adjust grading for different proof methods.