

# Computer Science Foundation Exam

May 6, 2016

Section II A

## DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,  
and you must work entirely on your own.**

### **SOLUTION**

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Logic)	7	
3	15	PRF (Sets)	10	
4	10	NTH (Number Theory)	7	
ALL	50		34	

**You must do all 4 problems in this section of the exam.**

**Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.**

1) (15 pts) PRF (Induction)

Use strong induction to prove that, for every non-negative integer,  $n$ ,  $F_n$  is even if and only if 3 divides  $n$ . Here  $F_n$  is the Fibonacci sequence given by the recurrence:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

Let  $P(n)$  be the open statement  $F_n$  is even if and only if 3 divides  $n$ .

**Basis Step:**

Case  $P(0)$ :

$F_0 = 0$  and 0 is even. We also know  $3 \mid 0$ .

Case  $P(1)$ :

$F_1 = 1$  and 1 is odd. We also know  $3 \nmid 1$ .

**Inductive Hypothesis:**

Assume for some arbitrary positive integer  $k$ , that for all  $0 \leq j \leq k$ ,  $P(j)$  is true. That is:

$$F_j \text{ is even} \Leftrightarrow 3 \mid j$$

**Inductive Step:** Show that  $P(k + 1)$  follows:

We must show two directions  $3 \mid k + 1 \Rightarrow F_{k+1}$  is even and  $F_{k+1}$  is even  $\Rightarrow 3 \mid k + 1$ . The second case we will show through the contrapositive.

Case ( $3 \mid k + 1$ ): In this case  $3 \nmid k - 1$  and  $3 \nmid k$ . From our inductive hypothesis  $F_{k-1}$  and  $F_k$  are both odd. The sum of two odd integers is even. Thus,  $F_{k+1}$  is even.

Case ( $3 \nmid k + 1$ ): In this case either  $3 \mid k$  or  $3 \mid k - 1$  but not both. From our inductive hypothesis either  $F_{k-1}$  or  $F_k$  must be odd and the other even. The sum of an odd integer and an even integer is odd. Thus,  $F_{k+1}$  is odd.

As we have shown  $3 \mid k + 1 \Rightarrow F_{k+1}$  is even and  $3 \nmid k + 1 \Rightarrow F_{k+1}$  is odd, it holds that  $3 \mid k + 1 \Leftrightarrow F_{k+1}$  is even or in other words  $P(k + 1)$  is true.

By the principle of mathematical induction,  $P(n)$  is true for all non-negative integers,  $n$ .

**Note: An alternative solution that breaks down  $F_{k+1}$  to a combination of  $F_{k-1}$  and  $F_{k-2}$  requires 3 base cases instead of 2.**

**Grading: Base case - 3 pts, IH - 2 pts, IS - 2 pts, Case ( $3 \mid (k+1)$ ) - 4 pts, Case ( $3 \nmid k + 1$ ) - 4 pts.**

2) (10 pts) PRF (Logic)

Here we have discovered some “rules of inference” that aren’t valid. Invalidate them by finding counter-examples that make each premise true but make the conclusion false.

$$(a) \frac{p \vee q \quad \neg p \vee \neg q}{\therefore p}$$

Let  $p = F$ ,  $q = T$ . This is the only counterexample that works. In this case, both premises are true (first is true due to  $q$ , second is true due to  $p$ ), and the conclusion is false.

**Grading: 4 pts, all or nothing, no explanation necessary.**

$$(b) \frac{(p \rightarrow q) \rightarrow r \quad \neg r}{\therefore \neg p}$$

Let  $p = T$ ,  $q = F$ ,  $r = F$ . This is also the only counterexample possible. In this case, the first premise is true because the left hand side is false, which makes the whole premise true. The second premise is true since  $r$  is false. Finally, the conclusion is false since  $p$  is true.

**Grading: 4 pts, all or nothing, no explanation necessary. Note - an alternative solution takes the two premises and uses them to prove that  $p$  must follow. This solution ALSO earns full credit.**

$$(c) \frac{p \rightarrow (q \rightarrow r) \quad \neg r}{\therefore p}$$

Let  $p = F$ ,  $q = T/F$ ,  $r = F$ . Here  $p$  and  $r$  must be false but  $q$  can be either for a counterexample.

**Grading: 2 pts, all or nothing, no explanation necessary.**

3) (15 pts) PRF (Sets)

Show for finite sets  $A, B, C$  that if  $B \cup C \subseteq A$  and  $A \times B \subseteq A \times C$ , then  $B \subseteq C$ .

We must show an arbitrary element of  $B$  must also be in  $C$ . Let  $b \in B$  arbitrarily. This element  $b$  must also be an element of  $B \cup C$ . As  $b \in B \cup C$  and  $B \cup C \subseteq A$ , it follows that  $b \in A$ .

From Cartesian product's definition, since  $b \in A$  and  $b \in B$ ,  $(b, b) \in A \times B$ . Now it follows from  $A \times B \subseteq A \times C$  that  $(b, b) \in A \times C$ . This means that  $b \in A$  and  $b \in C$ . Completing our proof since we've shown that  $b \in C$ .

QED

**Grading: 3 pts for stating what needs to be proved and starting with a premise of an element in B. 3 pts for concluding that the element is in A. 2 pts for concluding that (b,b) is in A x B. 2 pts for finishing the proof.**

Disprove for finite sets  $A, B, C$  that if  $A \times B \subseteq A \times C$ , then  $B \subseteq C$ .

Let  $A = \emptyset$ ,  $B = \{1, 2\}$ ,  $C = \{2\}$ . In this case, both  $A \times B$  and  $A \times C$  are the empty set since  $A$  is the empty set, making the premise true, but  $B$  isn't a subset of  $C$  since  $B$  contains 1 and  $C$  doesn't.

**Grading: 4 pts counter-example, 1 pt explanation**

**4) (10 pts) NTH (Number Theory)**

Prove that for any two distinct primes  $a$  and  $b$  there exists some prime  $c$  distinct from both  $a$  and  $b$  such that  $c \mid (a + b)$ .

There are two cases we must consider:

Case 1:

Suppose that  $a$  and  $b$  are both odd primes. As  $a$  and  $b$  are both odd,  $a + b$  must be even. This means the prime number 2 divides  $a + b$ . As 2 equals neither  $a$  nor  $b$  we have found a  $c$  meeting our criteria.

Case 2:

There is only one even prime, 2. Let  $a = 2$  without loss of generality and  $b$  be some other odd prime. We will show that  $2 + b$  is divisible by some prime neither 2 nor  $b$  by contradiction.

Suppose 2 and  $b$  are the only prime divisors of  $2 + b$ . The number  $2 + b$  is odd as 2 is even and  $b$  is odd. Thus 2 is not a divisor of  $2 + b$ . We also know that  $b$  does not divide  $2 + b$  by the division algorithm as  $2 + b = (1)b + 2$ ,  $0 \leq 2 < b$ . But this contradicts that  $2 + b$  must be representable by the product of primes by the fundamental theorem of arithmetic. Therefore, some other prime  $p$  must divide  $2 + b$ . Alternatively, we can argue that  $(2 + b) \equiv 2 \pmod{b}$  because  $b$  is greater than 2, proving that  $b$  isn't a divisor of  $(2 + b)$ .

QED

Alternate proof: Since  $a \neq b$ , it follows that  $a$  doesn't divide evenly into  $a+b$  and  $b$  doesn't divide evenly into  $a+b$ , since  $a \nmid b$  and  $b \nmid a$ . (A slightly more formal proof is showing that  $a+b \equiv a \pmod{b}$  and since  $a$  and  $b$  are distinct primes,  $a \not\equiv 0 \pmod{b}$ , and vice versa.)

By the Fundamental Theorem of Arithmetic, it follows that  $a+b$  has a unique prime factorization, and must have at least one prime factor. Since that prime factor isn't  $a$  or  $b$ , a distinct prime factor  $c$  must exist.

**Grading: Notes - there are other ways to prove these. Criteria for this solution:**

**5 pts even case, 5 pts odd case**

**Alternate criteria solution - 5 pts explanation that  $a$  and  $b$  don't divide  $a+b$ , 5 pts for concluding that a different prime must.**

# Computer Science Foundation Exam

May 6, 2016

## Section II B

### DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,  
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### **SOLUTION**

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	10	PRB (Probability)	7	
3	15	PRF (Functions)	10	
4	10	PRF (Relations)	7	
ALL	50		34	

**You must do all 4 problems in this section of the exam.**

**Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.**

## 1) (15 pts) CTG (Counting)

Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions. Justify your answers.

(a) (5 pts) If seven people are to be seated in a single row with ten chairs, how many unique arrangements are possible? (For example, if the people are labeled  $P_1$  through  $P_7$  and we use B to denote an empty chair, then one possible arrangement is  $P_1, B, B, P_2, P_4, P_3, P_7, P_5, B, P_6$ . Notice that the people are distinguishable but the empty chairs aren't.)

One process for counting the possibilities (where our guests are "A" through "G") is:

- Seat Guest A – 10 choices
- Seat Guest B – 9 choices
- Seat Guest C – 8 choices
- Seat Guest D – 7 choices
- Seat Guest E – 6 choices
- Seat Guest F – 5 choices
- Seat Guest G – 4 choices

Naturally, all the remaining seats are left empty. By the rule of product, the number of unique arrangements is  $\frac{10!}{3!}$ . **(Grading - 2 pt product, 3 pts terms)**

(b) (10 pts) How many of the arrangements from part (a) guarantee that none of the empty chairs are adjacent to one another?

Let  $x_1$  equal the number of chairs left of the first empty chair,  $x_2$  equal the number of chairs in between the first two empty chairs,  $x_3$  be the number of empty chairs in between the last two empty chairs and  $x_4$  equal the number of chairs to the right of the last empty chair. The number of non-negative integer solutions to the equation below is the number of ways to choose the empty chairs:

$$x_1 + x_2 + x_3 + x_4 = 7$$

such that  $x_2 > 0$  and  $x_3 > 0$ . We can deal with these restrictions by simply setting up new non-negative integer variables  $x_2 = x'_2 + 1$  and  $x_3 = x'_3 + 1$ , where  $x'_2 \geq 0$  and  $x'_3 \geq 0$ . Thus, now we want to find the number of non-negative integer solutions to:

$$\begin{aligned} x_1 + x'_2 + 1 + x'_3 + 1 + x_4 &= 7 \\ x_1 + x'_2 + x'_3 + x_4 &= 5 \end{aligned}$$

Using the combinations with repetition formula, we find that the number of non-negative solutions to the equation above is  $\binom{5+4-1}{4-1} = \binom{8}{3}$ .

Finally, the 7 people can be placed in the 7 non-empty chairs in  $7!$  ways, so the final answer is  $\binom{8}{3}7!$ .

**Grading: 7 pts for # of ways to choose empty chairs (many ways to do this, give partial as you see fit), 2 pts for seating the people, 1 pt for multiplying the two.**

## 2) (10 pts) PRB (Probability)

A mad scientist has created a probability-driven lock that can be permanently affixed to all manner of things (gym bags, lockers, vehicles, door locks, and so on). The lock has a five-digit LED display and a single button, and works as follows:

Once the lock is closed, if you want to open it, you press the button, and it generates a random sequence of five digits on its display. (Each digit is a random integer on the range zero through nine.) If the product of all five digits is odd, the lock opens. Otherwise, you have to press the button again to generate another random sequence of digits.

If you press the button three times without getting the lock to open, it seals itself shut forever.

If you put one of these locks on something, what is the probability that you'll actually be able to get it to open again?

The key insight here is that if any one of the five digits is even, then you have a factor of two in your overall product, and therefore the product is even. In order for the product to be odd, all five digits must be odd.

Since each of the five digits can either be even or odd (each with equal probability), we can consider the size of the sample space for a single press of the lock's button to be  $2^5 = 32$ . There is only one way for the product to be odd (that is, for all the digits to be even), so the probability of an odd product is  $\frac{1}{32} = 0.03125$ . This means the probability of an even product is  $\frac{31}{32}$ .

Another way to conceive of the sample space is to look at all possible five-digit numbers that could pop up. There are  $10^5 = 100,000$  such numbers. The number of outcomes where all five digits are odd is  $5^5 = 3,125$ , since there are five odd digits in the range zero through nine. So, the probability of an odd product is  $\frac{3,125}{100,000} = 0.03125$ .

The probability of getting three even products in a row is  $\left(\frac{31}{32}\right)^3$ . That also corresponds to the probability that the lock will be sealed shut forever. So, the probability that you'll actually succeed in opening the lock is  $1 - \left(\frac{31}{32}\right)^3 \approx 9.085\%$ .

**Alternate Solution:** Another way to arrive at that probability is to realize that you could open the lock on the first, second, or third try. These mutually disjoint events give rise to the following sum of probabilities:

$$\left(\frac{1}{32}\right) + \left(\frac{31}{32}\right)\left(\frac{1}{32}\right) + \left(\frac{31}{32}\right)\left(\frac{31}{32}\right)\left(\frac{1}{32}\right) \approx 9.085\%$$

**Grading:**

3 pts for realizing that the product is odd only if all five digits are odd.

3 pts for the cardinality of the sample space.

4 pts for their approach to the final answer (event complement or sum of probabilities)

*Please award partial credit for each piece as appropriate.*



## 3) (15 pts) PRF (Functions)

(a) (4 pts) Define finite sets  $A$  and  $B$  that satisfy *all three* of the following criteria simultaneously:

1.  $A$  and  $B$  are non-empty.
2. There exists a surjective function from  $A$  to  $B$ .
3. Every possible surjective function from  $A$  to  $B$  is also injective.

For this question, any sets  $A$  and  $B$  that are both non-empty (2 pts) and have the same cardinality (2 pts) will work.

(b) (4 pts) Define finite sets  $A$  and  $B$  that satisfy *all three* of the following criteria simultaneously:

1.  $A$  and  $B$  are non-empty.
2. There exists an injective function from  $A$  to  $B$ .
3. It is **impossible** to define an injective function from  $A$  to  $B$  that is also surjective.

For this question, any sets  $A$  and  $B$  that are both non-empty (2 pts) and where  $|A| < |B|$  (2 pts) will work.

(c) (4 pts) Define finite sets  $A$  and  $B$  that satisfy *all three* of the following criteria simultaneously:

1.  $A$  and  $B$  are non-empty.
2. There exists an injective function from  $A$  to  $B$ .
3. It is **possible** to define an injective function from  $A$  to  $B$  that is also surjective.

For this question, any sets  $A$  and  $B$  that are both non-empty (2 pts) and where  $|A| = |B|$  (2 pts) will work.

(d) (3 pts) Define finite sets  $A$  and  $B$  that satisfy *both* of the following criteria simultaneously:

1.  $A$  and  $B$  are non-empty.
2. It is **impossible** to define a function from  $A$  to  $B$  that is **not** injective.

For this question,  $A$  must contain a **single** element (2 pts), and  $B$  must be non-empty (1 pt).

## 4) (10 pts) PRF (Relations)

(a) (3 pts) What three properties must a relation satisfy in order to be a partial ordering relation?

It must be reflexive, transitive, and antisymmetric.

**Grading:**

-1 pt for each incorrect property listed

-1 pt for each correct property missing from the list

(b) (7 pts) Consider the relation  $\mathcal{R}$  on  $\mathbb{Z}^+$  defined as follows: For all positive integers  $x$  and  $y$ ,  $(x, y) \in \mathcal{R}$  if  $y$  is divisible by  $x$ . Prove or disprove that  $\mathcal{R}$  is a partial ordering relation.

 **$\mathcal{R}$  is reflexive:**

$x \mid x$  for all positive integers  $x$ , so  $(x, x) \in \mathcal{R}$ . Therefore,  $\mathcal{R}$  is reflexive.

 **$\mathcal{R}$  is transitive:**

If  $(x, y) \in \mathcal{R}$  and  $(y, z) \in \mathcal{R}$ , then we know  $x \mid y$  and  $y \mid z$ . It follows that  $y = xm$  for some integer  $m$ , and  $z = yn$  for some integer  $n$ . Thus,  $z = (xm)n = x(mn)$ , and since  $mn$  is an integer, we have that  $x \mid z$ . Therefore,  $(x, z) \in \mathcal{R}$ .

 **$\mathcal{R}$  is antisymmetric:**

If  $(x, y) \in \mathcal{R}$  and  $(y, x) \in \mathcal{R}$ , we know  $x \mid y$  and  $y \mid x$ . Due to the former, there exists a positive integer  $c$  such that  $y = cx$ . Due to the latter, there exists a positive integer  $d$  such that  $x = dy$ . Substituting the second equation into the first, we find that  $y = cx = c(dy)$ , so  $y = cdy$ . Since  $y$  must be non-zero, we get  $cd = 1$ . Since  $c$  and  $d$  are positive, it follows that both must be 1 and  $x$  and  $y$  are equal. Therefore,  $\mathcal{R}$  is antisymmetric.

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**Grading:**

1 pts for reflexive

3 pts for transitive

3 pts for antisymmetric

*If they try to prove (or disprove) the wrong properties, try to award at least half credit for the properties they worked with.*