Computer Science Foundation Exam

May 8, 2015

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	10	PRB (Probability)	6	
3	10	PRF (Functions)	6	
4	15	PRF (Relations)	10	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) CTG (Counting)

Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions. Justify your answers.

(a) (5 pts) A mouse always moves one unit in the positive x-axis or positive y-axis. In how many different ways can this mouse move from (0, 0) to (5, 7)?

Any unique path of the mouse can be expressed as a string of X's and Y's. If the starting point is (0, 0) and then ending point is (5, 7), then any string with 5 X's and 7 Y's will express movements between the two points. There are $\frac{12!}{5!7!}$ such strings, using the permutation formula (for string with repeated letters). Alternatively, we can choose 5 of the 12 possible letter slots to place our X's. This can be done in $\binom{12}{5}$ ways. (Note: Both answers as well as $\binom{12}{7}$ are equal to one another.) **Grading: 4 pts correct answer, 1 pt explanation, if there's multiplicative over or undercounting, 2 or 3 pts total, otherwise 0.**

(b) (10 pts) A classroom has 20 chairs set up in 4 rows of 5. Row number i ($1 \le i \le 4$) has chairs numbered 5i - 4 to 5i, from left to right, with row 1 being the front of the class. Stephen has a bad habit of trying to cheat from Amanda, Marty, and Kumar. If any of these three students is sitting to Stephen's left, or right, on either his own row or the row in front of him, then he's likely to cheat, since he can see papers on these desks. In how many ways can we place these four students in the classroom so that Stephen can't look at any of the other three students' papers? Count two placements as being different if at least one specific student is sitting in a different numbered chair in the two placements. (Note: The answer is a long sum of products. Leave your answer in this form, do not simplify.)

If Stephen is in the front row at either corner, there is only one seat that the three other students can't be placed, leaving 18 valid seats to place them. If Stephen is in one of the middles three seats in the front row, there are two seats the other three students can't be placed, leaving 17 valid seats to place them. If Stephen is on the second, third or fourth rows, on either the first or last chair in the row, then once again, the other three sutdents can't be placed in two seats, leaving 17 valid seats for them. Finally, if the students are in one of the remaining 9 seats, there are four forbidden seats for the other three, leaving 15 valid seats for them. This table sums up the relevant information:

Stephen's Seat Number	Locations for Stephen	#Valid Seats other 3
1,5	2	18
2,3,4,6,10,11,15,16,20	9	17
7,8,9,12,13,14,17,18,19	9	15

We can place the three students in N valid seats in N(N-1)(N-2) ways. Thus, to get our final count, we sum over the three categories of seats Stephen could be sitting in to get:

$$2 \times 18 \times 17 \times 16 + 9 \times 17 \times 16 \times 15 + 9 \times 15 \times 14 \times 13$$

Grading: 3 points for each case, 1 pt for adding the results from each case.

2) (10 pts) PRB (Probability)

(a) (5 pts) Consider the following algorithm to determine if an array, arr, of size n (indexes 0 through n-1, inclusive) is sorted from smallest to largest:

1) Repeat k times:
   i) Choose a random number, i, in between 0 and n-2, inclusive.
   ii) If arr[i] > arr[i+1], then answer that the array is out of order.
2) Answer that the array is in order.

Consider running this algorithm on the following array of size 10:

i	0	1	2	3	4	5	6	7	8	9
arr[i]	3	1	5	8	12	11	13	17	22	19

If we choose to run the algorithm with k = 5, what is the probability that the algorithm erroneously tells us that the array is sorted? Leave your answer as a fraction in lowest terms.

There are three locations out of nine where consecutive items are out of order in this array, indexes 0 and 1, indexes 4 and 5, and indexes 8 and 9. Thus, we have a $\frac{3}{9} = \frac{1}{3}$ chance on a single repetition of discovering that this array is not sorted and a $\frac{2}{3}$ chance on a single repetition that it is not discovered as being out of order. **(Grading - 3 pts)** For the algorithm to fail with k = 5, the latter would have to happen 5 times in a row. Assuming independence of the random number chosen in step 1i, the probability that this algorithm tells us the array is sorted is $(\frac{2}{3})^5 = \frac{32}{243}$. **(Grading - 2 pts)**

(b) (5 pts) In a search of a database with n elements, $\frac{1}{n}$ of all searches take n steps, $\frac{1}{2}$ of all searches take 2 steps, and the remaining $\frac{1}{2} - \frac{1}{n}$ of all searches take $\sqrt{n}$ steps. What is the expected number of steps a search of this database will take?

Using the definition of expectation, sum over the product of each probability and corresponding number of steps:

$$\frac{1}{n} \times n + \frac{1}{2} \times 2 + \left(\frac{1}{2} - \frac{1}{n}\right)\sqrt{n} = 1 + 1 + \frac{\sqrt{n}}{2} - \frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{2} + 2 - \frac{1}{\sqrt{n}}$$

Note: This expectation is dominated by the searches that take $\sqrt{n}$ steps, since these situations occur frequently. Thus, our expected number of steps is $\theta(\sqrt{n})$.

Grading: 3 pts for writing out definition, 2 pts for algebra. No need to mention theta bound.

3) (10 pts) PRF (Functions)

Let a function f: A → B be an injection and a function g: B → C be a surjection. Provide four examples to show that it is possible that the composition function, $g^{o}f$ might be

(a) neither injective, nor surjective
(b) injective, but not surjective
(c) surjective, but not injective
(d) both surjective and injective

For each example, specify the sets A, B and C as well as the functions f, g and $g^{o}f$.

(a) Let A = {1, 2}, B = {3, 4, 5}, C = {6, 7}, f = {(1, 3), (2, 4)} and g = {(3,6), (4, 6), (5,7)}. Then we have $g^{o}f$ = {(1, 6), (2, 6)}. This function isn't an injection since g(f(1)) = g(f(2)). This function isn't surjective since there is no value for which g(f(x)) = 7.

(b) Let A = {1, 2}, B = {3, 4, 5}, C = {6, 7, 8}, f = {(1, 3), (2, 4)} and g = {(3,6), (4, 7), (5,8)}. Then we have $g^{o}f$ = {(1, 6), (2, 7)}. This function is an injection since g(f(1)) ≠ g(f(2)). This function isn't surjective since there is no value for which g(f(x)) = 8.

(c) Let A = {1, 2}, B = {3, 4}, C = {5}, f = {(1, 3), (2, 4)} and g = {(3,5), (4, 5)}. Then we have $g^{o}f$ = {(1, 5), (2, 5)}. This function isn't an injection since g(f(1)) = g(f(2)). This function is surjective since there exists a value x for which g(f(x)) = 5, the only item in the set C.

(d) Let A = {1, 2}, B = {3, 4}, C = {5,6}, f = {(1, 3), (2, 4)} and g = {(3,5), (4, 6)}. Then we have $g^{o}f$ = {(1, 5), (2, 6)}. This function is an injection since g(f(1)) ≠ g(f(2)). This function is surjective since there exists a value x for which g(f(x)) = 5, and there exists a value x such that g(f(x)) = 6, and 5 and 6 are the only values in the set C.

Grading: 3 pts for (a) and (b), 2 pts for (c) and (d), grader decides partial

4) (15 pts) PRF (Relations)

Let R, S and T be relations and A, B and C be finite sets with $R \subseteq A \times B$, $S \subseteq A \times B$, and $T \subseteq B \times C$. Prove that $T \circ R \cup T \circ S = T \circ (R \cup S)$. (Note, we define the compositions of two relations $S \circ R$ to mean applying R, followed by applying S.)

To prove the two are equal, we must show that the left-hand side is a subset of the right-hand side and vice versa.

We will first show that $T \circ R \cup T \circ S \subseteq T \circ (R \cup S)$.

Let $(x, y) \in T \circ R \cup T \circ S$. We must show that $(x, y) \in T \circ (R \cup S)$. There are two cases to consider, by definition of union: (1) $(x, y) \in T \circ R$ and (2) $(x, y) \in T \circ S$ **(Grading – 2 pts)**

Let's consider the first case: $(x, y) \in T \circ R$

By definition of relation composition, there exists an element $z \in B$, such that $(x, z) \in R$ and $(z, y) \in T$. By definition of subset, since $(x, z) \in R$, it follows that $(x, z) \in R \cup S$. Thus, we can conclude that $(x, y) \in T \circ (R \cup S)$, since there exists an element $z \in B$ such that $(x, z) \in R \cup S$ and $(z, y) \in T$. **(Grading – 3 pts)**

Now we consider the second case: $(x, y) \in T \circ S$.

By definition of relation composition, there exists an element $z \in B$, such that $(x, z) \in S$ and $(z, y) \in T$. By definition of subset, since $(x, z) \in S$, it follows that $(x, z) \in R \cup S$. Thus, we can conclude that $(x, y) \in T \circ (R \cup S)$, since there exists an element $z \in B$ such that $(x, z) \in R \cup S$ and $(z, y) \in T$. **(Grading – 3 pts)**

Now, we prove the other direction, that $T \circ (R \cup S) \subseteq T \circ R \cup T \circ S$. Let $(x, y) \in T \circ (R \cup S)$. We must show that $(x, y) \in T \circ R \cup T \circ S$. **(Grading – 1 pt)**

By definition, there must exist $z \in B$, such that $(x, z) \in R \cup S$ and $(z, y) \in T$. **(Grading – 1 pt)**

By definition of union we have $(x, z) \in R$ or $(x, z) \in S$. **(Grading – 2 pts)**

Utilizing the definition of relation composition, in the former case we conclude that $(x, y) \in T \circ R$. In the latter case we conclude that $(x, y) \in T \circ S$. **(Grading – 3 pts)**

Thus, one of these two conclusions must hold. By definition of union, we conclude that $(x, y) \in T \circ R \cup T \circ S$. **(Grading – 1 pt)**

Grading note: The two proofs are reversible, so if students indicate that every step is an iff step clearly, they can show both directions by writing out each step once. This indication must be very clear to get credit though.