Computer Science Foundation Exam

May 8, 2015

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used,
and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	15	PRF (Logic)	10	
3	10	PRF (Sets)	6	
4	10	NTH (Number Theory)	6	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF (Induction)

Use strong induction to prove that, for every positive integer $n \geq 5$,

$$\exists x, y \in \mathbb{Z}^+ (n = 2^x + 3y) \vee (3 \mid n)$$

Let P(n) be an open statement equal to the proposition above.

Base Case (n=5): $5 = 2^1 + 3(1)$, when x=1, y=1
Base Case (n=6): $3 \mid 3 \rightarrow 3 \mid 3(2)$ by the multiplicative divisibility law $\therefore 3 \mid 6$
Base Case (n=7): $7 = 2^2 + 3(1)$, when x=2, y=1

Grading: 3 pts – 1 pt for each base case.

Inductive Hypothesis:

Assume $P(j)$ is true for all $j, 5 \leq j \leq k$ for some arbitrary k. In other words, assume the following: $\exists x, y \in \mathbb{Z}^+ (j = 2^x + 3y) \vee (3 \mid j)$

Grading: 2 pts – 1 pt for assuming the claim is true for one value, 1 pt for the accurate Strong induction assumption.

Inductive step (show true for n=k+1) **Grading: 2 pts**
We will handle the inductive step by cases:

We know that either $\exists x, y \in \mathbb{Z}^+ (j = 2^x + 3y)$ or $3 \mid j$ is true for all $5 \leq j < k + 1$ by our inductive hypothesis.

Case 1 $(3 \mid k - 2)$ **Grading: 4 pts**
We know $3 \mid 3$ by self-divisibility of non-zero integers and $3 \mid k - 2$ by the inductive hypothesis. By the additive law of divisibility $3 \mid (k - 2 + 3) \equiv 3 \mid (k + 1)$. Showing the P(k+1) true for this case.

Case 2 $\exists x, y \in \mathbb{Z}^+ \ (k - 2 = \ 2^x + 3y)$ **Grading: 4 pts**
Using existential instantiation we know there is a specific x and y where this statement holds by the Inductive Hypothesis.

$$k - 2 = 2^x + 3y$$
$$k - 2 + 3 = 2^x + 3y + 3$$
$$k + 1 = 2^x + 3(y + 1)$$

Because y+1 is a positive integer under closure of addition of positive integers proving this case holds for n=k+1. As at least one of the cases must be true by logical or this completes our proof by cases of the inductive step. By the strong induction argument our original conjecture holds.

$$Q.E.D.$$

2) (15 pts) PRF (Logic)

Consider the following sequence of positive integers where '?' represents undetermined values:

$$1, ?, 2, ?$$

The puzzle requires that any pair of adjacent values in the sequence must contain exactly 1 odd value. Your task is to replace each '?' with a positive integer. Formalize this problem as propositional logic (don't use quantifiers). You may use the following operator (XOR) to save space: $(x \oplus y) = (x \vee y) \wedge (\bar{x} \vee \bar{y})$.

Let p_i be true if the i-th element of our sequence is odd. (The propositional variable is false when p_i is even)

The puzzle can be modeled as the following compound proposition:

$$(p_1 \oplus p_2) \wedge (p_2 \oplus p_3) \wedge (p_3 \oplus p_4) \wedge p_1 \wedge \overline{p_3}$$

Grading: 1 pt per each clause

Use the Rules of Inference, Laws of Logic, and Substitution Rules to show a solution doesn't exist to this puzzle. Clearly state which rule you use for each step.

1. $p_1 \oplus p_2$	Premise
2. $p_2 \oplus p_3$	Premise
3. p_1	Premise
4. $\overline{p_3}$	Premise
5. $(p_1 \vee p_2) \wedge (\overline{p_1} \vee \overline{p_2})$	Def. of $\oplus$ on (1)
6. $(p_2 \vee p_3) \wedge (\overline{p_2} \vee \overline{p_3})$	Def. of $\oplus$ on (2)
7. $\overline{p_1} \vee \overline{p_2}$	Conjunctive Simplification on (5)
8. $\overline{p_2}$	Disjunctive Syllogism on (3) and (7)
9. $p_2 \vee p_3$	Conjunctive Simplification on (6)
10. p_3	Disjunctive Syllogism on (8) and (9)
11. $p_3 \wedge \overline{p_3}$	Conjunction on (4) and (10)
12. F	Inverse Laws on (11)

Deriving a contradiction means no solution exists.

Grading: 1 pt per step, except the premises, ½ pt per premise.

3) (10 pts) PRF (Sets)

Prove the following statement is true:

Let A, B and C be three finite sets. Prove that $(A \subseteq B \wedge A \subseteq C) \Rightarrow (A = \emptyset \vee (B \cap C \neq \emptyset))$.

Use proof by contradiction.

Assume that $\overline{(A = \emptyset \vee (B \cap C \neq \emptyset))}$ holds. This is equivalent to $A \neq \emptyset \wedge (B \cap C = \emptyset)$ by DeMorgan's Law. **(Grading: 2 pts)**

We know that for some arbitrary element x in A, x must also exist in both B and C by the definition of subset and the premise. This element must exist because A is not the empty set. **(Grading: 3 pts)**

If x is in both B and C then it must also exist in $B \cap C$ by definition of set intersection. This contradicts the fact that $B \cap C = \emptyset$. Therefore our original conjecture must hold. **(Grading: 2 pts)**

$$Q.E.D.$$

Show why $A = \emptyset$ must be in the conclusion via a single counter-example that disproves the following statement: $(A \subseteq B \wedge A \subseteq C) \Rightarrow (B \cap C \neq \emptyset)$.

Let $A = \emptyset, B = \{2\}, C = \{3\}$.

$A \subseteq B \wedge A \subseteq C$ is true while $B \cap C = \emptyset$. Our premise is true and our conclusion is false, breaking implication.

(Grading: 3 pts – all or nothing)

4) (10 pts) NTH (Number Theory)

Prove for arbitrary positive integer p, q that if p is prime and $\gcd(p, q) > 1$ then $p \leq q$.

Let $g = \gcd(p, q)$, we know from the definition of gcd that $g \mid p \wedge g \mid q$ and that g is the largest such integer. **(Grading - 2 pts)** We also know that $g > 1$ by the premise. Since p is a prime, it is only divisible by 1 and itself by definition of prime. If $g \mid p$ then we know $g = 1 \vee g = p$. **(Grading - 3 pts)** We know $g \neq 1$ because it is greater than 1. Therefore $g = p$. **(Grading - 2 pts)**

By algebraic substitution it follows that $p \mid q$. We know for positive integers a, b that $a \mid b \Rightarrow a \leq b$. Therefore, $p \leq q$ is true completing our direct proof. **(Grading - 3 pts)**