Computer Science Foundation Exam

May 8, 2015

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

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You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.
1) (15 pts) PRF (Induction)

Use strong induction to prove that, for every positive integer $n \geq 5$,

$$\exists x, y \in \mathbb{Z}^+(n = 2^x + 3y) \lor (3 \mid n)$$

Let $P(n)$ be an open statement equal to the proposition above.

*Base Case (n=5):* $5 = 2^1 + 3(1)$, when $x=1$, $y=1$

*Base Case (n=6):* $3 \mid 3 \rightarrow 3 \mid 3(2)$ by the multiplicative divisibility law $\div 3 \mid 6$

*Base Case (n=7):* $7 = 2^2 + 3(1)$, when $x=2$, $y=1$

**Grading:** 3 pts – 1 pt for each base case.

*Inductive Hypothesis:*

Assume $P(j)$ is true for all $j, 5 \leq j \leq k$ for some arbitrary $k$. In other words, assume the following: $\exists x, y \in \mathbb{Z}^+(j = 2^x + 3y) \lor (3 \mid j)$

**Grading:** 2 pts – 1 pt for assuming the claim is true for one value, 1 pt for the accurate strong induction assumption.

*Inductive step (show true for n=k+1)*

We will handle the inductive step by cases:

We know that either $\exists x, y \in \mathbb{Z}^+(j = 2^x + 3y)$ or $3 \mid j$ is true for all $5 \leq j < k + 1$ by our inductive hypothesis.

**Case 1 (3 \mid k - 2)**

We know $3 \mid 3$ by self-divisibility of non-zero integers and $3 \mid k - 2$ by the inductive hypothesis. By the additive law of divisibility $3 \mid (k - 2 + 3) \equiv 3 \mid (k + 1)$. Showing the $P(k+1)$ true for this case.

**Case 2 $\exists x, y \in \mathbb{Z}^+ (k - 2 = 2^x + 3y)$**

Using existential instantiation we know there is a specific $x$ and $y$ where this statement holds by the Inductive Hypothesis.

$$k - 2 = 2^x + 3y$$
$$k - 2 + 3 = 2^x + 3y + 3$$
$$k + 1 = 2^x + 3(y + 1)$$

Because $y+1$ is a positive integer under closure of addition of positive integers proving this case holds for $n=k+1$. As at least one of the cases must be true by logical or this completes our proof by cases of the inductive step. By the strong induction argument our original conjecture holds.

*Q.E.D.*
2) (15 pts) PRF (Logic)

Consider the following sequence of positive integers where ‘?’ represents undetermined values:

1, ?, 2, ?

The puzzle requires that any pair of adjacent values in the sequence must contain exactly 1 odd value. Your task is to replace each ‘?’ with a positive integer. Formalize this problem as propositional logic (don’t use quantifiers). You may use the following operator (XOR) to save space:

\[(x \oplus y) = (x \lor y) \land (\overline{x} \lor \overline{y}).\]

Let \(p_i\) be true if the \(i\)-th element of our sequence is odd. (The propositional variable is false when \(p_i\) is even)

The puzzle can be modeled as the following compound proposition:

\[(p_1 \oplus p_2) \land (p_2 \oplus p_3) \land (p_3 \oplus p_4) \land p_1 \land \overline{p_3}\]

**Grading: 1 pt per each clause**

Use the Rules of Inference, Laws of Logic, and Substitution Rules to show a solution doesn’t exist to this puzzle. Clearly state which rule you use for each step.

1. \(p_1 \oplus p_2\)  
   Premise  
2. \(p_2 \oplus p_3\)  
   Premise  
3. \(p_1\)  
   Premise  
4. \(\overline{p_3}\)  
   Premise  
5. \((p_1 \lor p_2) \land (\overline{p_1} \lor \overline{p_2})\)  
   Def. of \(\oplus\) on (1)  
6. \((p_2 \lor p_3) \land (\overline{p_2} \lor \overline{p_3})\)  
   Def. of \(\oplus\) on (2)  
7. \(\overline{p_1} \lor \overline{p_2}\)  
   Conjunctive Simplification on (5)  
8. \(\overline{p_2}\)  
   Disjunctive Syllogism on (3) and (7)  
9. \(p_2 \lor p_3\)  
   Conjunctive Simplification on (6)  
10. \(p_3\)  
   Disjunctive Syllogism on (8) and (9)  
11. \(p_3 \land \overline{p_3}\)  
   Conjunction on (4) and (10)  
12. \(F\)  
   Inverse Laws on (11)

Deriving a contradiction means no solution exists.

**Grading: 1 pt per step, except the premises, \(1/2\) pt per premise.**
3) (10 pts) PRF (Sets)

Prove the following statement is true:

Let $A$, $B$ and $C$ be three finite sets. Prove that $(A \subseteq B \land A \subseteq C) \Rightarrow (A = \emptyset \lor (B \cap C \neq \emptyset))$.

Use proof by contradiction. Assume that $(A = \emptyset \lor (B \cap C \neq \emptyset))$ holds. This is equivalent to $A \neq \emptyset \land (B \cap C = \emptyset)$ by DeMorgan’s Law. (Grading: 2 pts)

We know that for some arbitrary element $x$ in $A$, $x$ must also exist in both $B$ and $C$ by the definition of subset and the premise. This element must exist because $A$ is not the empty set. (Grading: 3 pts)

If $x$ is in both $B$ and $C$ then it must also exist in $B \cap C$ by definition of set intersection. This contradicts the fact that $B \cap C = \emptyset$. Therefore our original conjecture must hold. (Grading: 2 pts)

$Q.E.D.$

Show why $A = \emptyset$ must be in the conclusion via a single counter-example that disproves the following statement: $(A \subseteq B \land A \subseteq C) \Rightarrow (B \cap C \neq \emptyset)$.

Let $A = \emptyset, B = \{2\}, C = \{3\}$.

$A \subseteq B \land A \subseteq C$ is true while $B \cap C = \emptyset$. Our premise is true and our conclusion is false, breaking implication.

(Grading: 3 pts – all or nothing)
4) (10 pts) NTH (Number Theory)

Prove for arbitrary positive integer $p, q$ that if $p$ is prime and $\gcd(p, q) > 1$ then $p \leq q$.

Let $g = \gcd(p, q)$. we know from the definition of $\gcd$ that $g \mid p \land g \mid q$ and that $g$ is the largest such integer. (Grading - 2 pts) We also know that $g > 1$ by the premise. Since $p$ is a prime, it is only divisible by 1 and itself by definition of prime. If $g \mid p$ then we know $g = 1 \lor g = p$. (Grading - 3 pts) We know $g \neq 1$ because it is greater than 1. Therefore $g = p$. (Grading - 2 pts)

By algebraic substitution it follows that $p \mid q$. We know for positive integers $a, b$ that $a \mid b \Rightarrow a \leq b$. Therefore, $p \leq q$ is true completing our direct proof. (Grading - 3 pts)