Computer Science Foundation Exam

May 4th, 2012

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	15	PRF (Relations)	10	
3	15	PRF (Functions)	10	
4	15	NTH (Number Theory)	10	
ALL	60	---	40	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.

1) (15 pts) CTG (Counting)

(a) (3 pts) How many different permutations of the letters {a, b, c, d, e, f, g, h} are there?

(b) (4 pts) How many permutations of {a, b, c, d, e, f, g, h} are there that don't contain the letters "bad" appearing consecutively?

(c) (8 pts) How many permutations of {a, b, c, d, e, f, g, h} are there that don't contain either the letters "bad" appearing consecutively or the letters "fech" appearing consecutively?

Note: Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for either question.

Solution

(a) 8! = 8 * 7 * 6 … * 1 **(3 pts)**

(b) To compute the number of permutations that contain, we can consider "bad" as a "super letter." So we have {c,e,f,g,h,bad} as our alphabet. Therefore the number of permutations which contain "bad" is 6! **(2 pts)**

Therefore, the number of permutations that do not contain "bad" is 8! - 6! **(2 pts)**

(c) Let S_{bad} denote the set of permutations that have "bad" occuring in them. Similarly let S_{fech} denote the set of permutations that have "fech" occurring in them.

$|S_{bad} \cup S_{fech}| = |S_{bad}| + |S_{fech}| - |S_{bad} \cap S_{fech}|$

The number of permutations that contain "bad" ($|S_{bad}|$) as we computed in the previous problem is 6!. Similarly considering "fech" as a "super letter", we get that the number of permutations that contain "fech" is 5! **(3pts)**

To compute permutations that contain both "bad" and "fech", we need to look at permutations of the set {g,bad,fechg}. Therefore, the number of permutations that contain both "bad" and "fech" is 3!

Substituting in the above formula we get

$|S_{bad} \cup S_{fech}| =$ 6! + 5! - 3! **(3 pts)**

We are interested in finding the number of permutations that do not contain either "bad" or "fech".

Number of permutations without "bad" or "fech" = 8! – (6! + 5! – 3!) **(2 pts)**

2) (15 pts) PRF (Relations)

(a) (10 pts) Let **R** be the relation defined on the set of integers **Z** where $(a, b) \in$ **R** if and only if $a - b < 5$. Determine, with proof, whether or not **R** is reflexive, irreflexive, symmetric, anti-symmetric and transitive.

(b) (5pts) Suppose R and S are relations on the set $A=\{a,b,c,d\}$, where $R = \{(a,b),(a,d),(b,c),(c,c),(d,a)\}$ and $S = \{(a,c),(b,d),(d,a)\}$. Construct $R \circ S$.

<u>Solution</u>

(a)

R is reflexive. To see this, note that for all values a, $a - a = 0 < 5$. Thus, $(a,a) \in R$, for all integers a.

R is not irreflexive because $(1, 1) \in R$, as noted above.

R is not symmetric. Note that $(2, 10) \in R$ because $2 - 10 = -8 < 5$, but that $(10, 2) \notin R$, since $10 - 2 = 8 \geq 5$.

R is not anti-symmetric because $(3, 4) \in R$ because $3 - 4 = -1 < 5$ and $(4, 3) \in R$, because $4 - 3 = 1 < 5$.

R is not transitive because $(9, 6) \in R$ and $(6, 3) \in R$, but $(9, 3) \notin R$. Namely, we have $9 - 6 = 3 < 5$ and $6 - 3 = 3 < 5$, but $9 - 3 = 6 \geq 5$.

Grading: 2 pts for each item – 1 pt for each answer(yes or no) and 1 pt for each reason

(b) Construct $R \circ S$. Ans: $\{(a,a),(a,d),(d,c)\}$. (Grimaldi Answer)
{(a,c), (b,a), (d,b), (d,d)} (Rosen Answer)

Grading – Please accept either answer because two different textbooks for COT 3100 disagree on the answer. 1 pt for each item correctly listed, and give full credit if the answer is completely correct.

3) (15 pts) PRF (Functions)

(a) (10 pts) Prove "a function from an n-element set to an n-element set is one-to-one if and only if it is onto".

(b) (5 pts) Let $f(x) = e^x$ and $g(x) = 3x^2 – 4x + 5$, where the domain for both functions is the set of real numbers. Determine f(g(x)) and g(f(x)).

<u>Solution</u>

(a)

(part i) "→"

If *f* is one-to-one, it takes *exactly n* distinct values. **(2 pts)**
Since the range has *only n* values, *f* must be onto. **(2 pts)**
Thus, a one-to-one function from an *n*-element set to an *n*-element set is onto. **(1 pt)**

(part ii) "←"

If *f* is onto, then *f* takes *n* distinct values because it maps onto a set of size *n*. **(2 pts)**
But in this case, we may conclude that because there are only *n* values of *x*, all the values of *f(x)* are different. **(2 pts)**
Therefore, *f* must be one-to-one. **(1 pt)**

(b) $f(g(x)) = e^{3x^2-4x+5}$, and $g(f(x)) = 3e^{2x} – 4e^x + 5$. **(2 pts for f(g(x)), 3 pts for g(f(x)))**

4) (15 pts) NTH (Number Theory)

(a) (5 pts) If 133A – 554M = 1, does this guarantee that A has a multiplicative inverse mod M? If so, what is it? If not, why not?

(b) (10 pts) Use the Euclidean Algorithm to find gcd(70,102).

Solution

(a) Yes, it is 133 mod M. **(2 pts – Yes, 3 pts - 133)**

Based on corollary "if a ∈ Z_n and x and y are integers such that ax+ny=1, then the multiplicative inverse of a in Z_n is x mod n", let x=133, y=554 here.

(b)

102=70(1)+32
70=32(2)+6
32=6(5)+2
6=2(3)+0

gcd(70,102)=
gcd(70,32) = **(2pts)**
gcd(32,6) = **(2pts)**
gcd(6,2) = **(2pts)**
gcd(2,0) = **(2pts)**
2 **(2pts)**