Computer Science Foundation Exam

May 6, 2011

Section II A

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used, and you must work entirely on your own.

Question #	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Sets)	6	
3	15	PRF (Logic)	10	
ALL	40	---	26	

You must do all 3 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF (Induction)

Let T(n) be a recurrence relation defined by: $T(1) = 2$,
$T(n) = 2nT(n-1)$, for $n > 1$.

Prove that for all positive integers, $T(n) = 2^n n!$.

<u>Solution</u>

Base Case: n = 1. LHS = $T(1) = 2$, RHS = $2^1 1! = 2$. Both sides are equal, satisfying the base case. **(2 pts)**

Inductive hypothesis: Assume for an arbitrary positive integer n = k that $T(k) = 2^k k!$. **(2 pts)**

Inductive Step: Prove for n = k+1 that $T(k+1) = 2^{k+1}(k+1)!$. **(3 pts)**

$T(k+1) = 2(k+1)T(k)$ **(2 pts)**
$= 2(k+1)2^k k!$, plugging in the IH **(2 pts)**
$= 2^{k+1}(k+1)k!$ **(2 pts)**
$= 2^{k+1}(k+1)!$ **(2 pts)**

This proves the inductive step and completes the proof.

2) (10 pts) PRF (Sets)

Let A, B, and C be sets such that $C \subseteq B$. Use appropriate set theoretic laws and theorems to prove that

$(A \cap \neg B) \cup (B \cap \neg C) = (A \cup B) \cap \neg C$.

Be sure to explain each step of your proof.

LHS = $(A \cup B) \cap (A \cup \neg C) \cap (\neg B \cup B) \cap (\neg B \cup \neg C)$**, by the distributive law (1 pt)**

= $(A \cup B) \cap (A \cup \neg C) \cap U \cap (\neg B \cup \neg C)$**, by the inverse law (1 pt)**

= $(A \cup B) \cap (A \cup \neg C) \cap (\neg B \cup \neg C)$**, by the identity law (1 pt)**

= $(A \cup B) \cap (A \cup \neg C) \cap \neg(B \cap C)$**, DeMorgan's law (2 pts)**

= $(A \cup B) \cap (A \cup \neg C) \cap \neg C$**, the assumption** $C \subset B$ **implies** $B \cap C = C$ ***(3 pts)***

= $(A \cup B) \cap \neg C$**, by the absorption law** $X \cap (X \cup Y) = X$ ***(2 pts)***

= RHS

Note: There are other ways to solve this problem. Take off 1 or 2 points per mistake for other solutions.

3) (15 pts) PRF (Logic)

Prove the following logical expression is a tautology using the laws of logic equivalence and the definition of the conditional statement only. Show each step and state which rule is being used. (Note: You may combine both associative and commutative in a single step, so long as you do so properly.)

$$((\bar{p} \vee q) \wedge (p \vee r)) \rightarrow (q \vee r)$$

$((\bar{p} \vee q) \wedge (p \vee r)) \rightarrow (q \vee r)$ ↔ (Original)
$\overline{((\bar{p} \vee q) \wedge (p \vee r))} \vee (q \vee r)$ ↔ (Definition of Implication)
$(\overline{(\bar{p} \vee q)} \vee \overline{(p \vee r)}) \vee (q \vee r)$ ↔ (DeMorgan's Law)
$((\bar{\bar{p}} \wedge \bar{q}) \vee (\bar{p} \wedge \bar{r})) \vee (q \vee r)$ ↔ (DeMorgan's Law)
$((p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{r})) \vee (q \vee r)$ ↔ (Double Negation)
$((p \wedge \bar{q}) \vee q) \vee ((\bar{p} \wedge \bar{r}) \vee r)$ ↔ (Commutative and Associative)
$((q \vee \bar{q}) \wedge (q \vee p)) \vee ((r \vee \bar{r}) \wedge (r \vee \bar{p}))$ ↔ (Distributive Law)
$(T \wedge (q \vee p)) \vee (T \wedge (r \vee \bar{p}))$ ↔ (Inverse Law)
$(q \vee p) \vee (r \vee \bar{p})$ ↔ (Identity Law)
$(\bar{p} \vee p) \vee (r \vee q)$ ↔ (Commutative and Associative)
$T \vee (r \vee q)$ ↔ (Inverse Law)
T ↔ (Domination Law)

Grading: 1 pt off per mistake, ½ point off per rule used, rounding down.