You must do all 3 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.
1) (15 pts) PRF (Induction)

Let $T(n)$ be a recurrence relation defined by:

\[
T(1) = 2, \\
T(n) = 2nT(n - 1), \text{ for } n > 1.
\]

Prove that for all positive integers, $T(n) = 2^n n!$.

**Solution**

Base Case: $n = 1$. LHS = $T(1) = 2$, RHS = $2^1 1! = 2$. Both sides are equal, satisfying the base case. (2 pts)

Inductive hypothesis: Assume for an arbitrary positive integer $n = k$ that $T(k) = 2^k k!$. (2 pts)

Inductive Step: Prove for $n = k+1$ that $T(k + 1) = 2^{k+1} (k + 1)!$. (3 pts)

\[
T(k + 1) = 2(k + 1)T(k) \quad (2 \text{ pts})
\]

\[
= 2(k + 1)2^k k!, \text{ plugging in the IH (2 pts)}
\]

\[
= 2^{k+1} (k + 1)! \quad (2 \text{ pts})
\]

\[
= 2^{k+1} (k + 1)! \quad (2 \text{ pts})
\]

This proves the inductive step and completes the proof.
2) (10 pts) PRF (Sets)

Let $A$, $B$, and $C$ be sets such that $C \subseteq B$. Use appropriate set theoretic laws and theorems to prove that

$$(A \cap \neg B) \cup (B \cap \neg C) = (A \cup B) \cap \neg C.$$ 

Be sure to explain each step of your proof.

$LHS = (A \cup B) \cap (A \cup \neg C) \cap (\neg B \cup B) \cap (\neg B \cup \neg C)$, by the distributive law (1 pt)

$= (A \cup B) \cap (A \cup \neg C) \cap U \cap (\neg B \cup \neg C)$, by the inverse law (1 pt)

$= (A \cup B) \cap (A \cup \neg C) \cap (\neg B \cup \neg C)$, by the identity law (1 pt)

$= (A \cup B) \cap (A \cup \neg C) \cap \neg (B \cap C)$, DeMorgan’s law (2 pts)

$= (A \cup B) \cap (A \cup \neg C) \cap \neg C$, the assumption $C \subset B$ implies $B \cap C = C$ (3 pts)

$= (A \cup B) \cap \neg C$, by the absorption law $X \cap (X \cup Y) = X$ (2 pts)

$= RHS$

Note: There are other ways to solve this problem. Take off 1 or 2 points per mistake for other solutions.
3) (15 pts) PRF (Logic)

Prove the following logical expression is a tautology using the laws of logic equivalence and the definition of the conditional statement only. Show each step and state which rule is being used. (Note: You may combine both associative and commutative in a single step, so long as you do so properly.)

\[(\lnot p \lor q) \land (p \lor r) \rightarrow (q \lor r)\]

\[
\begin{align*}
((\lnot p \lor q) \land (p \lor r)) \rightarrow (q \lor r) & \iff \text{(Original)} \\
((\lnot p \lor q) \land (p \lor r)) \lor (q \lor r) & \iff \text{(Definition of Implication)} \\
((\lnot p \lor q) \lor (p \lor r)) \lor (q \lor r) & \iff \text{(DeMorgan’s Law)} \\
((\lnot p \land q) \lor (p \land \lnot r)) \lor (q \lor r) & \iff \text{(DeMorgan’s Law)} \\
((p \land q) \lor (p \land \lnot r)) \lor (q \lor r) & \iff \text{(Double Negation)} \\
((p \land q) \lor q) \lor ((p \land \lnot r) \lor r) & \iff \text{(Commutative and Associative)} \\
((q \land \overline{p}) \land (q \lor p)) \lor ((r \land \overline{p}) \land (r \lor \overline{p})) & \iff \text{(Distributive Law)} \\
(T \land (q \lor p)) \lor (T \land (r \lor \overline{p})) & \iff \text{(Inverse Law)} \\
(q \lor p) \lor (r \lor \overline{p}) & \iff \text{(Identity Law)} \\
(p \lor p) \lor (r \lor q) & \iff \text{(Commutative and Associative)} \\
T \lor (r \lor q) & \iff \text{(Inverse Law)} \\
T & \iff \text{(Domination Law)}
\end{align*}
\]

Grading: 1 pt off per mistake, ½ point off per rule used, rounding down.