Computer Science Foundation Exam

December 18, 2015

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

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You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.
1) (15 pts) CTG (Counting)

Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions. Justify your answers.

Consider a string \( s \) of lowercase latin letters of length 10.

(a) (5 pts) Suppose we require \( s \) to have no two consecutive letters that are the same. How many such strings \( s \) exist?

There are 26 ways to place the first character of \( s \). Each choice after that must be different than the previous character, so \( 25^9 \) ways to place remaining characters. Using rule of product, we derive \( 26 \cdot 25^9 \) total ways.

**Grading:** 2 pts first char, 2 pts other 9 chars, 1 pt mult terms

(b) (5 pts) Suppose we require \( s \) to have each consecutive triplet of letters be pairwise distinct. How many such strings \( s \) exist?

There are 26 ways to place the first character. We must differ from the first character with the second character, so this choice has 25 ways. The remaining 8 character must differ from the previous two characters so there are \( 24^8 \) ways to choose remaining characters. By rule of product, we derive \( 26 \cdot 25 \cdot 24^8 \) ways.

**Grading:** 1 pt first char, 1 pt second char, 2 pts other 9 chars, 1 pt mult terms

(c) (5 pts) Suppose we require \( s \) to have no two consecutive letters that are the same but additionally require that \( s \) be a palindrome. (A string that reads the same forward or backwards.) How many such strings \( s \) exist?

No two consecutive characters can be equal, but as \( s \) is an even length palindrome the middle two characters must be equal. This we have a contradiction in our requirements, resulting in 0 ways to form palindromes with the consecutive character requirement.

**Grading:** full credit for correct answer, 2 pts for answers that count # of arrangements of the first five letters, give other partial credit as you deem necessary
2) (15 pts) PRB (Probability)

Suppose we roll a 4 sided die with the numbers [1,4] written on them. After the first die roll we roll the die \( k \) times where \( k \) is the number on the first die roll. The number of points you score is the sum of the face-values on all die rolls (including the first). What is the expected number of points you will score?

Let \( E_r(k) \) be the expected score if you roll \( k \) dice. Suppose we have \( k \) dice. Let \( X_i \) be the discrete random variable representing the \( i^{th} \) die’s score. By linearity of expectations

\[
E(X_1 + X_2 + X_3 + X_4) = E(X_1) + E(X_2) + E(X_3) + E(X_4).
\]

As the die doesn’t change we can calculate \( E(X_i) \) using the standard formula for expected value:

\[
E(X_i) = 0.25(1) + 0.25(2) + 0.25(3) + 0.25(4) = \frac{10}{4}.
\]

Now we can calculate \( E_r(k) = k \cdot \frac{10}{4} \). (Grading: 5 pts, 1 per term and 1 for adding properly)

Let \( X \) be a discrete random variable representing the expected score total. We can calculate \( E(X) \) be calculating each expected value after knowing the outcome of the first die. This value is

\[
E(X) = 0.25(1 + E_r(1)) + 0.25(2 + E_r(2)) + 0.25(3 + E_r(3)) + 0.25(4 + E_r(4))
\]

\[
= 0.25 \left( 10 + \frac{10}{4} \right) = 0.25 \left( 10 + \frac{10}{4} \right) = \frac{35}{4} = 8.75
\]

There also exists a solution using probability trees. (Grading: 10 pts, 2 pts for each case and 2 pts for adding everything up properly)

Grading Note: Other solutions may exist, please try to award partial credit for significantly different approaches as you see fit.
3) (10 pts) PRF (Functions)

The function \( C(n, k) \) is known as the choose function.
Let \( X = \{(n, k)|n \in \mathbb{N} \land k \in \mathbb{N} \land k \leq n\} \). (Note that \( \mathbb{N} \) is the non-negative integers.)
For the purposes of this problem we define the choose function, \( C \), as follows:

\[
C: X \to \mathbb{Z}^+, C(n, k) = \frac{n!}{k!(n-k)!}
\]

(a) (5 pts) Is \( C \) injective? Prove or disprove this property.

No! We must show that \( f(a) = f(b) \not\Rightarrow a = b \). Let \( a = (3,2) \) and \( b = (3,1) \). It is clear that \( C(3,2) = C(3,1) \):

\[
C(3,2) = \frac{3!}{1!2!} = C(3,1)
\]
So \( f(a) = f(b) \). Yet \( (3,2) \neq (3,1) \).

Grading: 0 pts for answering yes, 2 pts for answering no, 3 pts for a specific example where \( f(a,b) = f(c, d) \) but either \( a \neq c \) or \( b \neq d \).

(b) (5 pts) Is \( C \) surjective? Prove or disprove this property.

Yes. To do this we must show that there exists \( (n, k) \) such that \( C(n, k) = z, z \in \mathbb{Z}^+ \) for all \( z \in \mathbb{Z}^+ \).

Let \( z \in \mathbb{Z}^+ \) arbitrarily. As \( \mathbb{Z}^+ \subseteq \mathbb{N} \), find \( C(z, 1) = \frac{z!}{(z-1)!1!} = \frac{z}{1} = z \).
As we have shown \( C(z, 1) = z \), we have found \( (n, k) \in \mathbb{N}^2 \) that hits each element of \( \mathbb{Z}^+ \).

QED

Grading: 0 pts for answering no, 2 pts for answering yes, 3 pts for showing input values and \( b \) such that for an arbitrary positive integer \( y \), \( f(a, b) = y \). You may give partial credit on this part as well.
4) (10 pts) PRF (Relations)

Consider the relation \( \mathcal{R} \) on \( \mathbb{Z}^2 \) where \( (a, b), (c, d) \) \( \in \mathcal{R} \) when \( ad - bc \geq 0 \). Determine (with proof) if \( \mathcal{R} \) meets or doesn’t meet each of these properties: reflexive, symmetric, anti-symmetric, transitive.

\( \mathcal{R} \) is reflexive: Show \( ((a, b), (a, b)) \in \mathcal{R} \). \( ab - ba = 0 \geq 0 \). Thus \( \mathcal{R} \) is reflexive.

\( \mathcal{R} \) is not symmetric: \( (3, 1), (1, 3) \) \( \in \mathcal{R} \) as \( 3 \cdot 3 - 1 \cdot 1 = 9 - 1 = 8 \geq 0 \), yet \( (1, 3), (3, 1) \) \( \notin \mathcal{R} \) as \( 1 \cdot 1 - 3 \cdot 3 = 1 - 9 = -9 < 0 \).

\( \mathcal{R} \) is not anti-symmetric: \( (1, 0), (-1, 0) \) \( \in \mathcal{R} \) as \( 1(0) - 0(-1) = 0 - 0 = 0 \geq 0 \).
(\( (-1, 0), (1, 0) \) \( \in \mathcal{R} \) as \( -1(0) - 0(1) = 0 - 0 = 0 \geq 0 \). Yet, \( (1, 0) \neq (-1, 0) \).

\( \mathcal{R} \) is not transitive:

\( (1, 0), (0, 1) \) \( \in \mathcal{R} \) as \( 1 \cdot 1 - 0 \cdot 0 = 1 - 0 = 1 \geq 0 \).
\( (0, 1), (0, -1) \) \( \in \mathcal{R} \) as \( 0(-1) - 1(0) = 0 - 0 \geq 0 \).

Yet \( (1, 0), (0, -1) \notin \mathcal{R} \) as \( 1(-1) - 0 \cdot 0 = -1 - 0 = -1 < 0 \).

In other words, sorting by the cross product is a bad idea.

Grading: 1 pt reflexive, 3 pts for each of the others, for the ones that are 3 pts 1 pt for saying no, 2 pts for finding a counter-example