Computer Science Foundation Exam

December 18, 2015

Section II B

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	15	PRB (Probability)	10	
3	10	PRF (Functions)	6	
4	10	PRF (Relations)	6	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) CTG (Counting)

Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions. Justify your answers.

Consider a string s of lowercase latin letters of length 10.

(a) (5 pts) Suppose we require s to have no two consecutive letters that are the same. How many such strings s exist?

There are 26 ways to place the first character of s. Each choice after that must be different than the previous character, so 25^9 ways to place remaining characters. Using rule of product, we derive $26 \cdot 25^9$ total ways.

Grading: 2 pts first char, 2 pts other 9 chars, 1 pt mult terms

(b) (5 pts) Suppose we require s to have each consecutive triplet of letters be pairwise distinct. How many such strings s exist?

There are 26 ways to place the first character. We must differ from the first character with the second character, so this choice has 25 ways. The remaining 8 character must differ from the previous two characters so there are 24^8 ways to choose remaining characters. By rule of product, we derive $26 \cdot 25 \cdot 24^8$ ways.

Grading: 1 pt first char, 1 pt second char, 2 pts other 9 chars, 1 pt mult terms

(c) (5 pts) Suppose we require s to have no two consecutive letters that are the same but additionally require that s be a palindrome. (A string that reads the same forward or backwards.) How many such strings s exist?

No two consecutive characters can be equal, but as s is an even length palindrome the middle two characters must be equal. This we have a contradiction in our requirements, resulting in 0 ways to form palindromes with the consecutive character requirement.

Grading: full credit for correct answer, 2 pts for answers that count # of arrangements of the first five letters, give other partial credit as you deem necessary

2) (15 pts) PRB (Probability)

Suppose we roll a 4 sided die with the numbers [1,4] written on them. After the first die roll we roll the die k times where k is the number on the first die roll. The number of points you score is the sum of the face-values on all die rolls (including the first). What is the expected number of points you will score?

Let $E_r(k)$ be the expected score if you roll k dice. Suppose we have k dice. Let X_i be the discrete random variable representing the i^{th} die's score. By linearity of expectations $E(X_1 + X_2 + X_3 + X_4) = E(X_1) + E(X_2) + E(X_3) + E(X_4)$. As the die doesn't change we can calculate $E(X_i)$ using the standard formula for expected value: $E(X_i) = 0.25(1) + 0.25(2) + 0.25(3) + 0.25(4) = \frac{10}{4}$. Now we can calculate $E_r(k) = k \cdot \frac{10}{4}$. **(Grading: 5 pts, 1 per term and 1 for adding properly)**

Let X be a discrete random variable representing the expected score total. We can calculate $E(X)$ be calculating each expected value after knowing the outcome of the first die. This value is $E(X) = 0.25\big(1 + E_r(1)\big) + 0.25\big(2 + E_r(2)\big) + 0.25\big(3 + E_r(3)\big) + 0.25\big(4 + E_r(4)\big)$

$$= 0.25\left(10 + \frac{10}{4}(1 + 2 + 3 + 4)\right) = 0.25\left(10\left(1 + \frac{10}{4}\right)\right) = \frac{35}{4} = 8.75$$

There also exists a solution using probability trees. **(Grading: 10 pts, 2 pts for each case and 2 pts for adding everything up properly)**

Grading Note: Other solutions may exist, please try to award partial credit for significantly different approaches as you see fit.

3) (10 pts) PRF (Functions)

The function $C(n,k)$ is known as the choose function.
Let $X = \{(n,k) | n \in \mathbb{N} \wedge k \in \mathbb{N} \wedge k \leq n\}$. (Note that $\mathbb{N}$ is the non-negative integers.)
For the purposes of this problem we define the choose function, C, as follows:

$$C: X \to \mathbb{Z}^+, C(n,k) = \frac{n!}{k!\,(n-k)!}$$

(a) (5 pts) Is C injective? Prove or disprove this property.

No! We must show that $f(a) = f(b) \nRightarrow a = b$. Let $a = (3,2)$ and $b = (3,1)$. It is clear that $C(3,2) = C(3,1)$:

$$C(3,2) = \frac{3!}{1!\,2!} = C(3,1)$$

So $f(a) = f(b)$. Yet $(3,2) \neq (3,1)$.

Grading: 0 pts for answering yes, 2 pts for answering no, 3 pts for a specific example where f(a,b) = f(c, d) but either a ≠ c or b ≠ d.

(b) (5 pts) Is C surjective? Prove or disprove this property.

Yes. To do this we must show that there exists (n,k) such that $C(n,k) = z, z \in \mathbb{Z}^+$ for all $z \in \mathbb{Z}^+$.

Let $z \in \mathbb{Z}^+$ arbitrarily. As $\mathbb{Z}^+ \subseteq \mathbb{N}$, find $C(z,1) = \frac{z!}{(z-1)!1!} = \frac{z}{1} = z$.
As we have shown $C(z,1) = z$, we have found $(n,k) \in \mathbb{N}^2$ that hits each element of $\mathbb{Z}^+$.

QED

Grading: 0 pts for answering no, 2 pts for answering yes, 3 pts for showing input values and b such that for an arbitrary positive integer y, f(a, b) = y. You may give partial credit on this part as well.

4) (10 pts) PRF (Relations)

Consider the relation $\mathcal{R}$ on $\mathbb{Z}^2$ where $\big((a,b),(c,d)\big) \in \mathcal{R}$ when $ad - bc \geq 0$. Determine (with proof) if $\mathcal{R}$ meets or doesn't meet each of these properties: reflexive, symmetric, anti-symmetric, transitive.

$\mathcal{R}$ is reflexive: Show $\big((a,b),(a,b)\big) \in \mathcal{R}$. $ab - ba = 0 \geq 0$. Thus $\mathcal{R}$ is reflexive.

$\mathcal{R}$ is not symmetric: $\big((3,1),(1,3)\big) \in \mathcal{R}$ as $3 \cdot 3 - 1 \cdot 1 = 9 - 1 = 8 \geq 0$, yet
$\big((1,3),(3,1)\big) \notin \mathcal{R}$ as $1 \cdot 1 - 3 \cdot 3 = 1 - 9 = -9 < 0$.

$\mathcal{R}$ is not anti-symmetric: $\big((1,0),(-1,0)\big) \in \mathcal{R}$ as $1(0) - 0(-1) = 0 - 0 = 0 \geq 0$.
$\big((-1,0),(1,0)\big) \in \mathcal{R}$ as $-1(0) - 0(1) = 0 - 0 = 0 \geq 0$.
Yet, $(1,0) \neq (-1,0)$.

$\mathcal{R}$ is not transitive:

$\big((1,0),(0,1)\big) \in \mathcal{R}$ as $1 \cdot 1 - 0 \cdot 0 = 1 - 0 = 1 \geq 0$.
$\big((0,1),(0,-1)\big) \in \mathcal{R}$ as $0(-1) - 1(0) = 0 - 0 \geq 0$.

Yet $\big((1,0),(0,-1)\big) \notin \mathcal{R}$ as $1(-1) - 0 \cdot 0 = -1 - 0 = -1 < 0$.

In other words, sorting by the cross product is a bad idea.

Grading: 1 pt reflexive, 3 pts for each of the others, for the ones that are 3 pts 1 pt for saying no, 2 pts for finding a counter-example