Computer Science Foundation Exam

December 18, 2015

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used,
and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	15	PRF (Logic)	10	
3	10	PRF (Sets)	6	
4	10	NTH (Number Theory)	6	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.

1) (15 pts) PRF (Induction)

Use mathematical induction to prove that, for every positive integer n,

$$5 \mid (8^n - 3^n)$$

Base Case (n = 1):
$8^1 - 3^1 = 5$, which is divisible by 5, so the base case holds. **(Grading: 2 pts)**

Inductive Hypothesis:
Assume for some arbitrary positive integer k, that $5 \mid (8^k - 3^k)$. That is to say, there exists some integer p such that $8^k - 3^k = 5p$. **(Grading: 2 pts)**

Inductive Step:
We want to show that $5 \mid (8^{k+1} - 3^{k+1})$. In other words, show that $8^{k+1} - 3^{k+1}$ can be written as $5q$, for some integer q. **(Grading: 2 pts)**

$8^{k+1} - 3^{k+1}$	$= (8)8^k - (3)3^k$	**(Grading: 2 pts)**
	$= (5+3)8^k - (3)3^k$	**(Grading: 2 pts)**
	$= (5)8^k + (3)8^k - (3)3^k$	**(Grading: 1 pts)**
	$= (5)8^k + (3)(8^k - 3^k)$	**(Grading: 1 pts)**
	$= (5)8^k + (3)(5p)$	by the inductive hypothesis **(Grading: 2 pts)**
	$= (5)(8^k + 3p)$	**(Grading: 1 pt)**
	$= (5)(q)$	where $q = 8^k + 3p$, which is an integer, since the set of integers is closed under multiplication and addition ($8^k \in \mathbb{Z}$ as $k \in \mathbb{Z}^+$)

So, $5 \mid (8^{k+1} - 3^{k+1})$.

Thus, by the principle of mathematical induction, $5 \mid (8^n - 3^n)$ for all positive integers, n. ■

2) (15 pts) PRF (Logic)

Validate the following argument using the laws of logic, substitution rules or rules of inference. List the rule used in each step and label the steps used in each derivation.

$$
\begin{array}{l}
(\neg q \vee n) \rightarrow r \\
q \\
z \rightarrow y \\
p \\
t \rightarrow \neg q \\
\hline
\therefore w \rightarrow ((p \vee \neg r) \wedge \neg t)
\end{array}
$$

Proof:

1.	$t \rightarrow \neg q$	Premise
2.	$q \rightarrow \neg t$	Contraposition (and Double Negation) on (2)
3.	q	Premise
4.	$\neg t$	Modus Ponens on (2) and (3)
5.	p	Premise
6.	$p \vee \neg r$	Disjunctive Amplification on (5)
7.	$(p \vee \neg r) \wedge \neg t$	Rule of Conjunction on (6) and (4)
8.	$\neg w \vee ((p \vee \neg r) \wedge \neg t)$	Disjunctive Amplification on (7)
9.	$w \rightarrow ((p \vee \neg r) \wedge \neg t)$	Definition of Implication (and Double Negation) on (9)

Grading: 5 pts for deriving not t, 3 pts for deriving p or not r, 3 pts for deriving (p or not r) and not t, 4 pts for deriving the final conclusion, take off a maximum of 4 pts for not listing reasons, taking off 1 pt per missing reason or incorrect reason stated

3) (10 pts) PRF (Sets)

a) Let $C = \{x, y\}$. What is $C \times C$?

$C \times C = \{(x, x), (x, y), (y, x), (y, y)\}$

b) Let $D = \{p, o, m\}$. What is $\mathcal{P}(D)$ (the power set of D)?

$\mathcal{P}(D) = \{\emptyset, \{p\}, \{o\}, \{m\}, \{p, o\}, \{p, m\}, \{o, m\}, \{p, o, m\}\}$

c) Let A and B be finite sets. Then, give the following in terms of $|A|$ and $|B|$:

$|A \times B| = |A| \cdot |B|$

$|\mathcal{P}(A)| = 2^{|A|}$

$|\mathcal{P}(A \times B)| = 2^{|A| \cdot |B|}$

d) Let A and B be finite sets. Give the following in terms of $|A|$, $|B|$, $|A \times B|$, and/or $|A \cap B|$. (Do not use $|A \cup B|$ in your answer.)

$|\mathcal{P}(A \cup B)| = 2^{|A|+|B|-|A \cap B|}$

a) Under what conditions does $A \times B$ equal to $B \times A$? (To receive full credit, your answer must cover all possible conditions where this occurs.)

Either A = B, or exactly one of A or B is the empty set. (The cross product of the empty set with any non-empty set is simply the empty set.)

(If both A and B are the empty set, then that falls under A = B.)

Grading for each part: 2 pts if completely correct, 1 pt if portions of the answer are correct, 0 pts otherwise, total is 10 pts for 5 parts

4) (10 pts) NTH (Number Theory)

Show that for all integers a and b, if $30 \mid (14a + 8b)$, then $30 \mid (2a - 46b)$.

There are a few ways to solve this one, but one is to observe that $2a$ differs from $30a$ by $28a$. From there, we have:

$30 \mid (14a + 8b)$, so $30 \mid 2(14a + 8b) = 28a + 16b$.

Clearly, $30 \mid 30a$. Since $30 \mid 30a$ and $30 \mid (28a + 16b)$, $30 \mid (30a - (28a + 16b)) = 2a - 16b$.

Also, $30 \mid 30b$. Since $30 \mid 30b$ and $30 \mid (2a - 16b)$, $30 \mid (2a - 16b - 30b) = 2a - 46b$. ■

Grading: 3 pts for trying some multiple of 14a + 8b, 2 pts for trying to add or subtract multiples of 30a, 2 pts for trying to add or subtract multiples of 30b, 3 pts for finishing up the problem.