Computer Science Foundation Exam

December 12, 2014

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used,
and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	15	PRB (Probability)	10	
3	10	PRF (Functions)	6	
4	10	PRF (Relations)	6	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) CTG (Counting)

Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions. Justify your answers.

(a) (7 pts) How many permutations of ELEPHANT do not have two consecutive vowels? (For example, you should count the permutation ELPHENAT, but not the permutation LEAPHENT.)

Use the consonants as separators: __ L __ P __ H __ N __ T __ **(1 pt)**

The underscores represent possible placement for the vowels. Since there are 3 vowels, their locations can be chosen in $\binom{6}{3}$ ways. **(2 pts)** Once these choices are made, the three vowels can be permuted in $\frac{3!}{2!1!}$, since there are 2 Es and 1 A. **(1 pt)** Finally, the consonants can be permuted in 5! ways (the picture above with the blanks is only 1 possible ordering of the consonants). **(2 pts)** Since each of these choices is independent of one another and each can be paired with any of the others, our total number of permutations of ELEPHANT with no two consonants in a row is $\binom{6}{3}\frac{3!}{2!1!} \times 5! = 7200$. **(1 pt)** (Note: There are many, many ways to solve this problem. Please map the points to different solution methods in a reasonable way.)

(b) (8 pts) Define a peak number to be a seven digit number, with unique digits such that the first four digits are in ascending order and the last four digits are in descending order. For example, 1357642 is a peak number. Note that a peak number may NOT start with the digit zero. How many peak numbers are there?

First, let's count peak numbers without the digit 0. We must choose our 7 digits (since they must be unique) from 9 possible digits, from 1 through 9. This can be done in $\binom{9}{7}$ ways. We know that the maximum of these values must be the digit in the middle. Then, we can choose any 3 of the 6 remaining to be the digits on the left in $\binom{6}{3}$ Once we make this choice, their ordering is fixed, as is the ordering of the digits on the right. Thus, there are $\binom{9}{7}\binom{6}{3}$ peak numbers without 0 as a digit. **(4 pts total - 2 pts 1st combo, 1 pt next, 1 pt mult)**

Now, consider peak numbers with 0 in them. The zero MUST occupy the units digit of the number. Thus, we only have the freedom to select 6 digits from the 9 remaining, which can be done in $\binom{9}{6}$ ways. Once these digits are chosen, the maximum digit is forced to be in the middle. This means we are free to choose the three digits on the left out of the five remaining, which we can do in $\binom{5}{3}$ ways. It follows that there are $\binom{9}{6}\binom{5}{3}$ peak numbers that contain the digit 0. **(3 pts - 1 pt each combo, 1 pt mult.)** Adding, we get a total of $\binom{9}{7}\binom{6}{3} + \binom{9}{6}\binom{5}{3} = 1560$. **(1 pt adding)**

2) (15 pts) PRB (Probability)

(a) (7 pts) It rains in Orlando on 30% of school mornings. If it rains, there is a 20% chance that Simone will be late for school. If it doesn't rain, there is a 5% chance she'll be late for school. Given that Simone goes to school in Orlando, what is the chance that she'll be late for school on a randomly selected school day?

We can add up the probability that it rains and she's late with the probability that it doesn't rain and she's late to get the total probabilty she'll be late. Let A be the event that it rains and B be the event that she's late. We have:

$$p(B) = p(A) \times p(B|A) + p(\bar{A}) \times p(B|\bar{A})$$
$$p(B) = .3 \times .2 + .7 \times .05 = .06 + .035 = .095$$

It follows that the chance she'll be late to school on a randomly chosen school day is 9.5%.

Grading: 3 pts probability tree breakdown, 3 pts substituting values, 1 pt answer.

(b) (8 pts) Jennifer is playing for a cash prize of $10,000 at halftime of a football game. To win the prize, she has to throw a football through the window of a car. She gets 10 throws and just one of them has to go through the car window for her to win. If none of them goes through, she wins nothing. If her chance of making an individual throw is 10%, what is her expected amount of winnings? Please leave your answer in powers, products, etc. and do not do any algebraic simplification.

The probability that none of them goes through is $.9^{10}$, since this would require her missing on 10 consecutive throws, each independent of one another**. (3 pts)** Thus, the probability that she hits at least one throw is $1 - .9^{10}$. **(3 pts)**

Finally, to calculate the expected value of the winnings, we multiply this probability by the cash prize to obtain $ $10000(1 - .9^{10})$. **(2 pts)**

Incidentally, this turns out to be roughly $6513.22.

3) (10 pts) PRF (Functions)

Let $f(x) = 3x^2 - 6x + 13$ with a domain of $x \leq 1$. Determine $f^{-1}(x)$ as well as its domain and range.

Substitute x for f(x) and y for x, and solve for y, which is $f^{-1}(x)$

$$x = 3y^2 - 6y + 13$$
$$x - 13 = 3(y^2 - 2y)$$
$$x - 13 = 3(y^2 - 2y + 1) - 3$$
$$x - 10 = 3(y - 1)^2$$
$$\frac{x - 10}{3} = (y - 1)^2$$
$$y - 1 = -\sqrt{\frac{x - 10}{3}}$$
$$f^{-1}(x) = 1 - \sqrt{\frac{x - 10}{3}}$$

In the second to last step, we take the negative of the square root, since we know that the domain of the original function is the range of the inverse function. Thus, since $y \leq 1$, it follows that $y - 1 \leq 0$. This dictates that we must take the negative of the square root, since a square root is defined to be non-negative for real numbers.

The domain for this function is $x \geq 10$, since the contents of the square root must be non-negative.

Grading: 1 pt for switching x and y, 1 pt for subbing 13 and factoring 3, 2 pts for completing the square, 1 pt for dividing by three, 2 pts for taking the negative of the square root, 1 pt for the inverse, 1 pt for the range, 1 pt for the domain

4) (10 pts) PRF (Relations)

Let R be the following relation over positive integers:

R = { (a, b) | a < 2b or a > 2b }

Determine, with proof, whether or not R is (a) reflexive, (b) symmetric, (c) transitive.

(a) R is reflexive. Note that for any arbitrary ordered pair of the form (a, a), since $a > 0$, $a < 2a$, satisfying the first clause of the definition for R. **(3 pts)**

(b) R is not symmetric. Consider the ordered pair (1, 2). $(1,2) \in R$, since $1 < 2(2)$, satisfying the first clause of the definition of R. But, $(2,1) \notin R$ because $2 < 2(1)$ is false AND $2 > 2(1)$ is also false**. (1 pt answer, 1 pt counter-example, 1 pt proof)**

(c) R is also not transitive. Using an idea similar to the previous counter-example, note that $(2,3) \in R$ since $2 < 2(3)$ and $(3,1) \in R$, since $3 > 2(1)$. But, as previously mentioned, $(2,1) \notin R$ because $2 < 2(1)$ is false AND $2 > 2(1)$ is also false. Thus, we have found values a, b and c such that $(a,b) \in R$, $(b,c) \in R$, and $(a,c) \notin R$, proving that R is NOT transitive. **(1 pt answer, 2 pts counter-example, 1 pt proof)**

Grading note: Before starting work on the problem, a student can note that this relation is equivalent to all ordered pairs where $a \neq 2b$. Then they can use this more intuitive description in their proofs.