Computer Science Foundation Exam

December 13, 2013

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	15	PRB (Probability)	10	
3	10	PRF (Functions)	6	
4	10	PRF (Relations)	6	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.

1) (15 pts) CTG (Counting)

Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions.

(a) (3 pts) An IP address is usually expressed as four numbers, separated by dots. Each of the four components can be expressed as an 8-bit binary number, but we usually write it in decimal for readability. For example, 172.16.254.1 is a valid IP address, since each component can be expressed as an 8-bit binary number.

Assuming that every such combination of 8-bit numbers is a valid address, how many valid IP addresses exist?

Solution There are $2^8 = 256$ eight-bit numbers. By the product rule, there are 256^4 valid IP addresses. **(Grading: 1 pt for 256 or 2^8, 1 pt for 4, 1 pt for exponent)**

(b) (5 pts) Movies and TV shows frequently use IP addresses as part of a technobabble scene, but they must be careful not to use a real address. An easy way to do this is to use numbers that are too large to fit the 8-bit restriction. For example, 132.513.151.319 is an invalid address, because two of its components (513 and 319) are too large.

We define a *fake address* as any such address where at least one component is too large. If no component is permitted to have more than three digits, how many fake IP addresses can we make?

Solution Each component can take 1000 possible values (0-999). Thus the total number of addresses is $1000^4 = 10^{12}$. Of these, we know that 256^4 are valid IP addresses. By complementarity, there are $10^{12} - 256^4$ fake addresses. **(Grading: 3 pts for total, 2 pts for subtraction, if they try w/o subtraction it's tough! Just give partial as you see fit. It's a complicated inclusion/exclusion otherwise…)**

(c) (7 pts) You have been assigned a chunk of 100 IP addresses, and you would like to distribute them equally among the 5 projects you are working on. Assuming that both addresses and projects are distinguishable, how many ways are there to do this?

Solution Standard application of the multinomial theorem gives $\frac{100!}{20!20!20!20!20!}$. Alternatively, we can express this as $\binom{100}{20}\binom{80}{20}\binom{60}{20}\binom{40}{20}\binom{20}{20}$, which simplifies to the same thing. **(Grading: multiple forms work here, 1 pt for each term, 2 pts for multiplying.)**

2) (15 pts) PRB (Probability)

(a) (4 pts) A single card is drawn from a standard deck of 52 playing cards. If the card is red, what is the probability that it is a face card (i.e., a Jack, Queen or King)?

Solution P(face card | red card) = P(red face card) / P(red card) = (6/52) / (26/52) = 3/13
(Grading: 2 pts for numerator, 2 pts for denominator, can be solved by ignoring the red part…)

(b) (4 pts) A card is drawn three times from a deck, with replacement. (In other words, we draw the card, note it down, and then replace it in the deck before we draw again.) Given that the first two cards were the same, what is the probability that the third card will NOT be the same as the first two?

Solution

$$P(\text{third different} \mid \text{first two same}) = \frac{P(\text{first two same and third different})}{P(\text{first two same})}$$

Now, we have $P(\text{first two same}) = \frac{(52\times 1)}{(52\times 52)} = \frac{1}{52}$
And, $P(\text{first two same and third different}) = \frac{52\times 1\times 51}{52\times 52\times 52} = \frac{51}{52^2}$

Plugging these in gives $P(\text{third different} \mid \text{first two same}) = 51/52$
Grading: since there is independence, you can just do p(third different) directly…2 pts for numerator, 2 pts for denominator

(c) (7 pts) Every year, you go on vacation. Four out of every five years, you go to an established vacation spot, and half of these vacations are fun. The remaining one year, you pick an obscure place, and three-quarters of these vacations are fun. If this year's vacation was not fun, what is the probability that you went to an obscure place this year?

Solution Let F be the event of having fun, and V be the event that we went to an established vacation spot. We want to find $P(\neg V|\neg F)$.

By Bayes theorem, we have:

$$P(\neg V|\neg F) = \frac{P(\neg F|\neg V)\cdot P(\neg V)}{P(\neg F|\neg V)\cdot P(\neg V) + P(\neg F|V)\cdot P(V)}$$

Plugging in the values (taking complements where we need to), we get:

$$P(\neg V|\neg F) = \frac{1/4\cdot 1/5}{1/4\cdot 1/5 + 1/2\cdot 4/5} = \frac{1}{9}$$

Grading: no independence here…2 pt realizing to use Bayes, 2 pts numerator, 4 pts denominator

3) (10 pts) PRF (Functions)

Let $f: A \to A$ be an injective function, where A is a **finite** set. Prove that f must be surjective.

Solution

Let $|A| = n$. For the sake of contradiction, assume that f is **not** surjective. Then there exists some $y \in A$ such that there is no x where $f(x) = y$, i.e., some element of A has no pre-image. **(3 pts)**

As a function, f is defined for all elements of A. But since y is off-limits, the range of A can have no more than $n - 1$ elements. By the pigeonhole principle, there must be some z such that there are two distinct elements x_1 and x_2 where $f(x_1) = f(x_2) = z$. **(5 pts)**

But this directly violates our hypothesis that f is injective, creating a contradiction. The result follows. **(2 pts)**

Note: There are many ways to do this, it's essentially a counting argument. Map the points accordingly.

4) (10 pts) PRF (Relations)

Let A be the set of all people, and B be the set of all historical events. We define $R \subseteq A \times B$ and $S \subseteq A \times B$ as follows:

R = { (a, b) | a has read a book about b }
S = { (a, b) | a has watched a documentary on b}

Consider the relation $T = S^{-1} \circ R$.

(i) Express T in set-builder notation.
(ii) Is T reflexive?
(iii) Is T symmetric?
(iv) Is T antisymmetric?
(v) Is T transitive?

Solution

(i) $T = \{(a, b) \mid a \text{ has read a book about some event that } b \text{ has watched a documentary about}\}$

(ii) T is not reflexive, since there is no guarantee that a person must have both read about and watched a documentary of some event.

(iii) T is not symmetric. Just because Harold read a book about the same event that that Joni watched a documentary about doesn't mean that Joni has ever read a book about any event for which Harold saw the documentary, though it is possible!

(iv) T is not anti-symmetric, since it possible for two people to have consumed both forms of media about some event. Or we can have (a, b) due to one event and (b, a) due to a second event. Either way, there's no reason for $a = b$ to follow.

(v) T is not transitive. (a, b) and (b, c) may both be in the relation, but connected by different events, meaning that (a, c) won't necessarily be present.

Grading: 2 pts each, give partial if you deem necessary.