Computer Science Foundation Exam

December 13, 2013

Section II A

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used, and you must work entirely on your own.

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Logic)	6	
3	10	PRF (Sets)	6	
4	15	NTH (Number Theory)	10	
ALL	50	---	32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF (Induction)

Use mathematical induction to prove the following statement is true for integers n > 1:

$$n! < n^n.$$

Solution

Base Case: n = 2
LHS: 2! = 2
RHS: $2^2 = 4$ **(2 pts)**

Because the LHS is less than the RHS the base case is shown to be true.

Inductive Hypothesis: Assume for an arbitrary positive integer n = k that $k! < k^k$. **(2 pts)**

Inductive Step: Prove for n = k+1 that $(k+1)! < (k+1)^{k+1}$. **(2 pts)**

(starting with the IH)

k!	$< k^k$
(k+1)k!	$< (k+1)\ k^k$
(k+1)!	$< (k+1)\ k^k$
	$< (k+1)\ (k+1)^k$
	$< (k+1)^{k+1}$

(starting with k+1)

(k+1)!	= (k+1)k!	**(2 pts)**
	$< (k+1)\ k^k$ by the IH	**(2 pts)**
	$< (k+1)\ (k+1)^k$	**(3 pts)**
	$< (k+1)^{k+1}$	**(2 pts)**

Thus, $n! < n^n$ for all integers n > 1.

2) (10 pts) PRF (Logic)

Use the Rules of Inference and the Law of Contraposition to validate the conclusion drawn below. (Each of the items above the dotted line is a premise, while the conclusion to draw is below the dotted line.) Show each step and state which rule is being used.

$q \rightarrow (u \wedge t)$
$u \rightarrow p$
q

$p \wedge t$

Solution

Step	**Rule**
1. $q \rightarrow (u \wedge t)$	**Premise**
2. q	**Premise**
3. $(u \wedge t)$	**Modus Ponens (1, 2)**
4. u	**Simplification (3)**
5. $u \rightarrow p$	**Premise**
6. p	**Modus ponens (4, 5)**
7. t	**Simplification (4)**
8. $p \wedge t$	**Conjunction (6, 7)**

Grading: 1 pt off for each mistake, if mostly correct, if fewer than 4 steps, 1 pt for each correct step.

3) (10 pts) PRF (Sets)

Let A, B and C be finite sets of integers. If A ∩ B = A ∩ C and $\bar{A} \cap B = \bar{A} \cap C$, prove that B = C.

Givens: A ∩ B = A ∩ C $\bar{A} \cap B = \bar{A} \cap C$

$(A \cap B) \cup (\bar{A} \cap B) = (A \cap C) \cup (\bar{A} \cap C)$	Hint/Definition of Union **(4 pts)**
$B \cap (A \cup \bar{A}) = C \cap (A \cup \bar{A})$	Distributive Law **(2 pts)**
$B \cap U = C \cap U$	Complement Law **(2 pts)**
B = C	Identity Law **(2 pts)**

Grading: If students use a different approach, try to map points as best as possible and stay consistent.

4) (15 pts) NTH (Number Theory)

Find an inverse of 19 modulo 141 using the extended Euclidean algorithm.

Hint (optional, can be removed): Recall that an inverse (a’) of a number means 19a’ ≡ 1 mod 141

(This question can easily be converted to: Express the gcd(19, 141) as a linear combination of 19 and 141 if we don’t want inverses.)

141 = 19 * 7 + 8
19 = 8 * 2 + 3
8 = 3 * 2 + 2
3 = 2 * 1 + 1
2 = 1 * 2 + 0 **Grading: 5 pts, 1 per line**

1 = 3 – 2
2 = 8 – 3*2
3 = 19 – 8*2
8 = 141 – 19*7

1 = 3 – (8 – 3*2)
1 = 3*3 – 8

1 = 3(19 – 8*2) – 8
1 = 3 * 19 – 7 * 8

1 = 3 * 19 – 7(141 – 19*7)
1 = 52 * 19 – 7 * 141 **Grading: 9 pts to get to here.**

a’ = 52 **Grading: 1 pt to extract the correct answer**