Computer Science Foundation Exam

December 14, 2012

Section II A

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used, and you must work entirely on your own.

<table>
<thead>
<tr>
<th>Question</th>
<th>Max Pts</th>
<th>Category</th>
<th>Passing</th>
<th>Score</th>
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</table>

You must do all 3 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.
1) (15 pts) PRF (Induction)

Recall that the Fibonacci sequence is defined as follows: \( F_0 = 0, F_1 = 1, \) and \( F_n = F_{n-1} + F_{n-2}, \) for all integers \( n > 1. \) Use induction to prove the following matrix exponentiation statement for all positive integers \( n: \)

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.
\]

**Solution**

Base Case: \( n = 1. \) Using the given definition of the Fibonacci Numbers, we find that \( F_0 = 0, F_1 = 1, \) and \( F_2 = 1. \) Thus, in evaluating the claim for \( n = 1, \) we find:

\[
\text{LHS} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \text{RHS} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Thus, the claim is true for \( n = 1. \) (2 pts)

Inductive Hypothesis: Assume for an arbitrary positive integer \( n = k \) that

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}. \quad (2 \text{ pts})
\]

Inductive Step: Prove for \( n = k+1 \) that

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}. \quad (2 \text{ pts})
\]

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k, \text{ by definition of exponents} \quad (2 \text{ pts})
\]

\[
= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}, \text{ using the inductive hypothesis} \quad (2 \text{ pts})
\]

\[
= \begin{pmatrix} F_{k+1} + F_k & F_k + F_{k-1} \\ F_{k+1} & F_k \end{pmatrix}, \text{ carrying out the matrix multiply} \quad (3 \text{ pts})
\]

\[
= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}, \text{ using the definition of Fibonacci numbers} \quad (2 \text{ pts})
\]
2) (10 pts) PRF (Logic)

Fill in the truth table below to prove that the following logical expression is a tautology.

\[(\bar{q} \land r) \rightarrow ((p \lor q) \rightarrow p)\]

**Solution**

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>(\bar{q} \land r)</th>
<th>(p \lor q)</th>
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<th>((\bar{q} \land r) \rightarrow ((p \lor q) \rightarrow p))</th>
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Grading: 3 pts for first column, 3 pts for second column, 3 pts for third column, 1 pt for last column. Give 3 pts if a column is completely correct, 2 pts if it's a majority correct and 1 pt if there are 1-4 entries correct, for the first three columns. Only award the point for the last column if it's completely correct.
3) (15 pts) PRF (Sets)

Let \( A, B \) and \( C \) be finite sets of integers. Prove or disprove the following assertions:

a) If \( C \subseteq A \cap B \), then \( A - B \subseteq A - C \).

b) If \( A \cap B \subseteq C \), then \( (A - C) \cup (B - C) \cup (A \cap B) = A \cup B \).

**Solution (a)**

This assertion is true. In order to prove it, we must show that for an arbitrarily chosen element \( x \), if \( x \in A - B \), then \( x \in A - C \). (2 pts)

If \( x \in A - B \), by definition of set difference, \( x \in A \) and \( x \notin B \). (1 pt)

We are given that \( C \subseteq A \cap B \). Since \( A \cap B \subseteq B \) by the definition of set intersection and the subset relation is transitive, we can ascertain that \( C \subseteq B \). (3 pts)

But, since \( x \notin B \), it follows that \( x \notin C \), because if \( x \in C \) it would contradict the assertion that \( C \subseteq B \). (3 pts)

(Another way to derive this is to note that \( C \subseteq B \) means that for any arbitrary element \( x \), that if \( x \in C \), then \( x \in B \). The contrapositive of this statement is: if \( x \notin B \), then \( x \notin C \). Finally, since we know that \( x \notin B \) already, applying the statement we just derived yields that \( x \notin C \).)

Finally, since we’ve shown for our arbitrarily chosen element \( x \) that \( x \in A \) and \( x \notin C \), it follows by definition of set difference that \( x \in A - C \). (1 pt)

**Solution (b)**

This assertion is false. In order to see this, consider the following counter-example:

\[
A = \{1\} \\
B = \{2\} \\
C = \{1, 2\} \text{ (3 pts for the counter example)}
\]

In this example, we have \( A \cap B = \emptyset \), thus it’s vacuously true that \( A \cap B \subseteq C \).

But, notice that each of the sets \( (A - C) \), \( (B - C) \), and \( (A \cap B) \) are empty. Thus, the left-hand side of the statement we are trying to prove evaluates to the empty set. But, \( A \cup B = \{1, 2\} \), which is not empty. (2 pts for showing it's a counter example)

Thus, in this particular example, the statement does not hold. Thus, it's not true for all finite sets of integers, \( A, B \) and \( C \).