Computer Science Foundation Exam

December 17, 2010

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

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You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.
4) (CTG) Counting (15 pts)

(a) (6 pts) Find the coefficient of $x^2y^3z^5$ in $(x + y + z)^{10}$.

(b) (9 pts) How many permutations of ENGINEERING do not contain consecutive vowels? For example, ENIGNERENIG should be counted, but EENIGNERING should not be counted.

Solution:

(a) \(\binom{10}{2}\binom{8}{3}\) or \(\frac{10!}{2!3!5!}\). There are other formats for the same answer. (1 pt for each number in either format and 2 pts for dividing or multiplying, depending on the format)

(b) The word ENGINEERING contains 5 vowels (3 Es, 2 Is) and 6 consonants (3 Ns, 2 Gs, 1 R). (2 points) There are \(\binom{5}{2}\) ways to arrange the vowels among themselves (2 points) and \(\binom{6}{3}\binom{3}{2}\) ways to arrange the consonants among themselves (2 points). To construct a valid configuration, first consider an arrangement of the consonants. Then, the 6 consonants creates 7 locations (or gaps) where we can place vowels. There are \(\binom{7}{5}\) ways to select a subset of locations for the vowels (2 points). The total number of configurations is \(\binom{5}{2}\binom{6}{3}\binom{3}{2}\binom{7}{5} = 12600\) (1 point). There are multiple ways to come to the same answer. Students may leave the answer in \(\binom{n}{r}\) format, exact numeric value is not required.
5) (PRF) Relations (15 pts)

Let $A \subseteq Z$ be a subset of integers and let $R$ be a relation over $A \times A$ defined as follows:

$$(x_1, x_2)R (y_1, y_2) \text{ whenever } x_1 y_2 \leq y_1 x_2 \text{ for } x_1, x_2, y_1, y_2 \in A$$

(a) (9 pts) Let $A = Z$, determine whether $R$ satisfies each of the following properties. Justify your answers.

(i) Reflexive

(ii) Symmetric

(iii) Transitive

(b) (6 pts) Let $A = Z^+$, determine whether or not $R$ is transitive. Justify your answer.

Solution:

(a) When $A = Z$,

(i) $R$ is reflexive. (1 point) $x_1 x_2 \leq x_1 x_2$, so $(x_1, x_2) R (x_1, x_2)$. (1 point)

(ii) $R$ is not symmetric. (1 point) $1 \times 2 \leq 3 \times 1$, so $((1, 1), (3, 2)) \in R$. But $((3, 2), (1, 1)) \notin R$ since $3 \times 1 > 1 \times 2$. (2 points) There are other examples.

(iii) $R$ is not transitive. (1 points) Let $(x_1, x_2) = (-1, -1)$, $(y_1, y_2) = (2, 3)$ and $(z_1, z_2) = (3, 2)$. (2 points) Then $((x_1, x_2), (y_1, y_2)) \in R$ since $(-1 \times 3) \leq (2 \times -1)$ and $((y_1, y_2), (z_1, z_2)) \in R$ since $(2 \times 2) \leq (3 \times 3)$. But $((x_1, x_2), (z_1, z_2)) \notin R$ since $(-1 \times 2) > (3 \times -1)$. (1 point) There are other examples.

(b) When $A = Z^+$, $R$ is transitive. (2 points)

Let $((x_1, x_2), (y_1, y_2)) \in R$ and $((y_1, y_2), (z_1, z_2)) \in R$, (1 point) then

$x_1 y_2 \leq y_1 x_2$ and $y_1 z_2 \leq z_1 y_2$ (1 point)

$$\frac{x_1}{x_2} \leq \frac{y_1}{y_2} \quad \text{and} \quad \frac{y_1}{y_2} \leq \frac{z_1}{z_2}$$

(This step is valid because $x_2, y_2, z_2 > 0$)

so

$$\frac{x_1}{x_2} \leq \frac{y_1}{y_2} \leq \frac{z_1}{z_2}, \quad (1 \ point)$$

$$\frac{x_1}{x_2} \leq \frac{z_1}{z_2} \rightarrow x_1 z_2 \leq z_1 x_2 \rightarrow ((x_1, x_2), (z_1, z_2)) \in R. \quad (1 \ point)$$

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6) (PRF) Functions (15 pts)

Let \( f : A \to B \) and \( g : B \to A \) be functions such that

(i) \( \forall y \in B, (f \circ g)(y) = y \)

(ii) \( \exists x \in A : (g \circ f)(x) \neq x \)

Prove that \( f \) is surjective (onto), but not injective (one-to-one).

Solution:

\( f \) is surjective:

We need to prove that \( \forall y \in B, \exists x \in A : f(x) = y. \) (2 points)

Let \( y \in B \) be arbitrary. Consider the element \( g(y) \). Let \( x = g(y) \in A. \) (3 points) Then,

\[ f(x) = f(g(y)) = y \] (by condition (i)) (2 points). So, for all element \( y \in B \), there exists \( x \in A \) (in particular \( x = g(y) \)) such that \( f(x) = y. \)

\( f \) is not injective:

We need to prove that \( \exists x_1, x_2 \in A : x_1 \neq x_2 \) and \( f(x_1) = f(x_2). \) (2 points)

By condition (ii), \( \exists x \in A : (g \circ f)(x) \neq x. \) Let \( x_1 = x \) and \( x_2 = (g \circ f)(x). \) (2 points) Note that \( x_1 \neq x_2. \) (1 point)

Then,

\[ f(x_2) = f \left( g \left( f(x) \right) \right) \] (by definition of \( x_2 \))

\[ = (f \circ g)(f(x)) \] (by definition of function composition)

\[ = f(x) \] (by condition (i))

\[ = f(x_1). \] (by definition of \( x_1 \)).

(3 points)

Thus, there exist \( x_1, x_2 \in A : x_1 \neq x_2 \) and \( f(x_1) = f(x_2). \)
7) (NTH) Number Theory (15 pts)

(a) (5 pts) What is the largest prime factor of 100!?

(b) (10 pts) Using the Extended Euclidean Algorithm, find at least one integer solution (for x and y) to the equation \(177x + 530y = 7\).

Solution:

(a) 97. The largest prime factor of 100! is the largest prime number less than or equal to 100.

(b) Apply Extended Euclidean Algorithm for 177 and 530.

\[
\begin{align*}
\text{GCD}(530, 177) &= 530 = 2(177) + 176 \\
= \text{GCD}(177, 176) &= 177 = 176 + 1 \\
= 1 & \\
\text{(3 points)}
\end{align*}
\]

So, \(1 = 177 - 176\) and \(176 = 530 - 2(177)\).

Substitute.

\[
\begin{align*}
1 &= 177 - (530 - 2(177)) = 3(177) - 530 \\
\text{(2 points)}
\end{align*}
\]

Multiply by 7.

\[
\begin{align*}
7 &= (21)(177) - 7(530). \\
\text{(2 points)}
\end{align*}
\]

So, \((177)(21) + 530)(-7) = 7\).

One possible answer is \(x = 21, y = -7\). (3 points)

(Note: There are other viable solutions.)