Computer Science Foundation Exam

December 17, 2010

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
4	15	CTG (Counting)	10	
5	15	PRF (Relations)	10	
6	15	PRF (Functions)	10	
7	15	NTH (Number Theory)	10	
ALL	60	---	40	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

4) (CTG) Counting (15 pts)

(a) (6 pts) Find the coefficient of $x^2y^3z^5$ in $(x + y + z)^{10}$.

(b) (9 pts) How many permutations of ENGINEERING do not contain consecutive vowels? For example, ENIGNERENIG should be counted, but EENGINERING should not be counted.

Solution:

(a) $\binom{10}{2}\binom{8}{3}$ or $\frac{10!}{2!3!5!}$. There are other formats for the same answer. **(1 pt for each number in either format and 2 pts for dividing or multiplying, depending on the format)**

(b) The word ENGINEERING contains 5 vowels (3 Es, 2 Is) and 6 consonants (3 Ns, 2 Gs, 1 R). **(2 points)** There are $\binom{5}{2}$ ways to arrange the vowls among themselves **(2 points)** and $\binom{6}{3}\binom{3}{2}$ ways to arrange the consonants among themselves **(2 points).** To construct a valid configuration, first consider an arrangement of the consonants. Then, the 6 consonants creates 7 locations (or gaps) where we can place vowels. There are $\binom{7}{5}$ ways to select a subset of locations for the vowels **(2 points).** The total number of configurations is $\binom{5}{2}\binom{6}{3}\binom{3}{2}\binom{7}{5} = 12600$ **(1 point).** There are multiple ways to come to the same answer. Students may leave the answer in $\binom{n}{r}$ format, exact numeric value is not required.

5) (PRF) Relations (15 pts)

Let $A \subseteq Z$ be a subset of integers and let R be a relation over $A \times A$ defined as follows:

$$(x_1, x_2) R\ (y_1, y_2) \text{ whenever } x_1 y_2 \leq y_1 x_2 \text{ for } x_1, x_2, y_1, y_2 \in A$$

(a) (9 pts) Let $A = Z$, determine whether R satisfies each of the following properties. Justify your answers.
  (i) Reflexive
  (ii) Symmetric
  (iii) Transitive

(b) (6 pts) Let $A = Z^+$, determine whether or not R is transitive. Justify your answer.

Solution:

(a) When $A = Z$,

(i) R is reflexive. **(1 point)** $x_1 x_2 \leq x_1 x_2$, so $(x_1, x_2)\ R\ (x_1, x_2)$. **(1 point)**

(ii) R is not symmetric. **(1 point)** $1 \times 2 \leq 3 \times 1$, so $\big((1,1),(3,2)\big) \in R$. But $\big((3,2),(1,1)\big) \notin R$ since $3 \times 1 > 1 \times 2$. **(2 points)** There are other examples.

(iii) R is not transitive**. (1 points)** Let $(x_1, x_2) = (-1,-1), (y_1, y_2) = (2,3)$ and $(z_1, z_2) = (3,2)$. **(2 points)** Then $\big((x_1, x_2),\ (y_1, y_2)\big) \in R$ since $(-1 \times 3) \leq (2 \times -1)$ and $\big((y_1, y_2),\ (z_1, z_2)\big) \in R$ since $(2 \times 2) \leq (3 \times 3)$. But $\big((x_1, x_2),\ (z_1, z_2)\big) \notin R$ since $(-1 \times 2) > (3 \times -1)$. **(1 point)** There are other examples.

(b) When $A = Z^+$, R is transitive. **(2 points)**

Let $\big((x_1, x_2),\ (y_1, y_2)\big) \in R$ and $\big((y_1, y_2),\ (z_1, z_2)\big) \in R$, **(1 point)** then

$x_1 y_2 \leq y_1 x_2$ and $y_1 z_2 \leq z_1 y_2$ **(1 point)**

$\frac{x_1}{x_2} \leq \frac{y_1}{y_2}$ and $\frac{y_1}{y_2} \leq \frac{z_1}{z_2}$ (This step is valid because $x_2, y_2, z_2 > 0$)

so $\frac{x_1}{x_2} \leq \frac{y_1}{y_2} \leq \frac{z_1}{z_2}$**, (1 point)**

$\frac{x_1}{x_2} \leq \frac{z_1}{z_2} \rightarrow x_1 z_2 \leq z_1 x_2 \rightarrow \big((x_1, x_2),\ (z_1, z_2)\big) \in R$. **(1 point)**

6) (PRF) Functions (15 pts)

Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be functions such that

(i) $\forall y \in B, (f \circ g)(y) = y$

(ii) $\exists\, x \in A : (g \circ f)(x) \neq x$

Prove that f is surjective (onto), but not injective (one-to-one).

Solution:

f is surjective:

We need to prove that $\forall y \in B, \exists\, x \in A : f(x) = y$. **(2 points)**

Let $y \in B$ be arbitrary. Consider the element $g(y)$. Let $x = g(y) \in A$. **(3 points)** Then, $f(x) = f(g(y)) = y$ (by condition (i)) **(2 points).** So, for all element $y \in B$, there exists $x \in A$ (in particular $x = g(y)$) such that $f(x) = y$.

f is not injective:

We need to prove that $\exists x_1, x_2 \in A : x_1 \neq x_2$ and $f(x_1) = f(x_2)$. **(2 points)**

By condition (ii), $\exists\, x \in A : (g \circ f)(x) \neq x$. Let $x_1 = x$ and $x_2 = (g \circ f)(x)$. **(2 points)** Note that $x_1 \neq x_2$. **(1 point)**

Then,

$f(x_2) = f\Big(g\big(f(x)\big)\Big)$ (by definition of x_2)

$= (f \circ g)(f(x))$ (by definition of function composition)

$= f(x)$ (by condition (i))

$= f(x_1)$. (by definition of x_1).

(3 points)

Thus, there exist $x_1, x_2 \in A : x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

7) (NTH) Number Theory (15 pts)

(a) (5 pts) What is the largest prime factor of 100!?

(b) (10 pts) Using the Extended Euclidean Algorithm, find at least one integer solution (for x and y) to the equation $177x + 530y = 7$.

Solution:

(a) 97. The largest prime factor of 100! is the largest prime number less than or equal to 100.

(b) Apply Extended Euclidean Algorithm for 177 and 530.

GCD(530, 177)	530 = 2 (177) + 176
= GCD(177, 176)	177 = 176 + 1
= 1	
(3 points)	

So, 1 = 177 – 176 and 176 = 530 – 2 (177).

Substitute.
1 = 177 – (530 – 2 (177)) = 3(177) – 530
(2 points)

Multiply by 7.
7 = (21)(177) – 7 (530).
(2 points)

So, (177)(21) + (530)(-7) = 7.

One possible answer is $x = 21, y = -7$. (3 points)

(Note: There are other viable solutions.)