Computer Science Foundation Exam

December 18, 2009

Section II B

DISCRETE STRUCTURES SOLUTIONS

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ______________________________

PID: ______________________________

In this section of the exam, there are four (4) problems.
You must do ALL of them.
Each counts for 15% of the Discrete Structures exam grade.
Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

Question	Max Pts	Category	Passing	Score
4	15	CTG (Counting)	10	
5	15	PRF (Relations)	10	
6	15	PRF (Functions)	10	
7	15	NTH (Number Theory)	10	
ALL	60	---	40	

4) (CTG) Counting (15 pts)

(a) (5 pts) How many permutations are there in the following DNA sequence "ATTTAGCCCCCATG"?

(b) (10 pts) There are four students and each owns one book. They put the four books in a box and randomly pick one. What is the probability that exactly two students get their own book? (Note: There are 24 possible assignments of books, so the probability is simply the number of different assignments where exactly two students get their own book divided by 24.)

(a) Using the permutation formula, we get $\frac{14!}{2!3!4!5!}$, since there are 2 Gs, 3 As, 4 Ts, 5 C in "ATTTAGCCCCCATG"**. Grading – 2 pts for the numerator, 1 pt for a fraction, and 2 pts for the denominator.**

(b) We can choose the two students who get assigned their own book in $\binom{4}{2}$ = 6 ways (4 pts). Since we must have EXACTLY two students get their own books, we are forced to make sure that the other two remaining students get each other's book (so there's no choice here, the books both receive are forced.) (4 pts)

Since there are 24 possible book assignments, the desired probability is 6/24 = ¼. (2 pts)

5) (PRF) Relations (15 pts)

Suppose A is the set composed of all ordered pairs of positive integers. Let R be the relation defined on A where (a,b) R (c,d) means that $a + d = b + c$.
(a) Prove that R is an equivalence relation. (12 pts)
(b) Find [(2,5)]. (3 pts)

(a) Reflexive: $a + b = b + a$; (a,b) R (a,b); **(3 pts)**
Symmetric: if $a + d = b + c$, then $c + b = d + a$; **(3 pts)**
Transitive: if $a + d = b + c$ and $c + f = d + e$, then $a + d - (d + e) = (b + c) - (c + f)$, therefore $a - e = b - f$, or $a + f = b + e$. **(6 pts)**
Conclusion R is an equivalence relation.
(b) [(2,5)] = {(1,4), (2,5), (3,6), (4,7), (5,8), (6,9), …}.
More formally, it's { (x,y) | x – y = 3 }

(3 pts – 2 pts for listing at least two ordered pairs in the list and a third point for indicating somehow that the list is infinite)

6) (PRF) Functions (15 pts)

Let f and g be functions such that f: A→B and g: B→C. Prove or disprove the following:

(a) (7 pts) If $g \circ f$ is injective, then g is injective.

(b) (8 pts) If f and g are surjective, then $g \circ f$ is surjective

(a) This is false. (2 pts) Consider the following counter-example:

A = {1} B = {2,3} C = {4}. (2 pts)
Let the function f = {(1,2)} and the function g = {(2,4), (3,4)}. (2 pts)

In this example, $g \circ f$ = {(1,4)} is injective (2 pt), but g is not injective (1 pt).

Grading: There are many, many examples that work. Give 2 points for stating the result. 4 points for fully writing out the example and 1 point for verifying that it is a counter-example.

(b)We can prove this through contradiction.

Assume the opposite, that $g \circ f$ is NOT surjective. (1 pt)

Then there exists some element z ∈ C such that there is NO element x ∈ A such that $g \circ f$ (x) = z. (2 pts)

Now, since g is surjective, this means that there is some element y ∈ B such that g(y) = z. (2 pts)

Since, f is surjective, this mean there is some elements x' ∈ A such that f(x') = y. Therefore g(f(x'))=z. (2 pt)

This contradicts the fact that there is no element y in A such that g(x) = z. Thus, it follows that the initial assume is incorrect and $g \circ f$ must be surjective. (1 pts)

Grading: Note, there are other ways to prove this, correspondingly award points.

7) (NTH) Number Theory (15 pts)

(a) (10 pts) Prove that for every positive integer x of exactly four digits, if the sum of digits is divisible by 3, then x itself is divisible by 3. (i.e. consider x = 6132, the sum of digits of x is 6+1+3+2 = 12, which is divisible by 3, so x is also divisible by 3).

(b) (5 pts) Compute the GCD(465,95).

(a) Let the digits of x be x_3, x_2, x_1, x_0, written from most-significant to least-significant digit. In this situation, we have that

$x = 10^3x_3 + 10^2x_2 + 10x_1 + x_0$ <u>(3 pts)</u>

We will prove a slightly stronger statement than the one we are being asked to prove:

$x \equiv x_3 + x_2 + x_1 + x_0 \pmod 3$ <u>(2 pts)</u>

$10^3x_3 + 10^2x_2 + 10x_1 + x_0 \equiv 1000x_3 + 100x_2 + 10x_1 + x_0 \pmod 3$

$\equiv 999x_3 + 99x_2 + 9x_1 + (x_3 + x_2 + x_1 + x_0) \pmod 3$ <u>(3 pts)</u>

$\equiv 0 + (x_3 + x_2 + x_1 + x_0) \pmod 3$, since each of these terms is divisible by 3.

$\equiv 0 + (x_3 + x_2 + x_1 + x_0) \pmod 3$, <u>(2 pts)</u>

(b) 465 = 4 x 95 + 85 (1 pt)
95 = 1 x 85 + 10 (1 pt)
85 = 8 x 10 + 5 (1 pt)
10 = 2 x 5 (1 pt)
GCD (465, 95) = 5 (1 pt)