Computer Science Foundation Exam

December 19, 2008

Section II A

DISCRETE STRUCTURES

KEY

Question #	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Sets)	6	
3	15	PRF (Logic)	10	
ALL	40	---	26	

PART A

1) (15 pts) PRF (Induction)

Using proof by induction, prove that $\sum_{i=1}^{n}[i(i+1)] = \frac{n(n+1)(n+2)}{3}$, for all positive integers *n*.

Solution.

Base case: n = 1. LHS = $\sum_{i=1}^{1}[i(i+1)] = 1(2) = 2$, RHS = $\frac{1(1+1)(1+2)}{3} = \frac{6}{3} = 2$. Thus LHS = RHS and the base case is proven. **(3 pts)**

Inductive hypothesis: Assume for an arbitrary positive integer $n = k$ that $\sum_{i=1}^{k}[i(i+1)] = \frac{k(k+1)(k+2)}{3}$. **(3 pts)**

Inductive Step: We will prove for $n = k+1$ that $\sum_{i=1}^{k+1}[i(i+1)] = \frac{(k+1)(k+2)(k+3)}{3}$ **(2 pts)**

$\sum_{i=1}^{k+1}[i(i+1)] = \sum_{i=1}^{k}[i(i+1)] + (k+1)(k+2)$ **(2 pts)**

$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$, using the inductive hypothesis **(2 pts)**

$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$ **(1 pt)**

$= \frac{(k+1)(k+2)(k+3)}{3}$. **(2 pts)**

The inductive step is complete. □

2) (10 pts) PRF **(**Sets)

Let A, B and C be arbitrary sets taken from the positive integers.

Prove the following statement: If $A - B \subseteq C$, then $A \subseteq B \cup C$.

Solution.

$A - B \subseteq C$ is the premise.
We assume $x \in A$. Our goal is to prove that $x \in B \cup C$. **(1 pt)**
There are two cases: either $x \in B$ or $x \in \overline{B}$. **(1 pt)**

<u>Case 1</u>
$x \in B$.
It follows that $x \in B$ or $x \in C$ by disjunctive amplification. **(1 pt)**
Thus, $x \in B \cup C$ by definition of union. **(1 pt)**

<u>Case 2</u>
$x \in \overline{B}$.
We know that $x \in A$, so $x \in A$ and $x \in \overline{B}$. **(1 pt)**
Thus, $x \in A - B$ by definition of set difference. **(1 pt)**
We are given that $A - B \subseteq C$.
Therefore $x \in C$ by definition of subset. **(1 pt)**
It follows that $x \in B$ or $x \in C$ by disjunctive amplification. **(1 pt)**
Hence $x \in B \cup C$ by definition of union. **(1 pt)**

In both cases, we have proven that $x \in B \cup C$.
Therefore, we conclude that $A \subseteq B \cup C$ by definition of subset. □ **(1 pt)**

Alternate Solution:

Use proof by contradiction. Assume to the contrary, that A is NOT a subset of $B \cup C$. It follows that there exists an element x such that $x \in A$ and $x \notin B \cup C$. **(2 pts)**

Logically, this means that $x \notin B$ AND $x \notin C$, since x can not be in either set, otherwise it would be contained in the union. (Formally, we have $x \notin B \cup C$ is equivalent to $\neg(x \in B \cup C) \leftrightarrow \neg(x \in B \vee x \in C) \leftrightarrow (\neg(x \in B) \wedge \neg(x \in C)) \leftrightarrow (x \notin B) \wedge (x \notin C)$.) **(4 pts)**

It follows by the definition of set difference that $x \in A - B$, since x is an element of A but not B. **(2 pts)** But $x \in A - B$ and $x \notin C$ contradicts the given information that $A - B \subseteq C$. **(2 pts)**

It follows that our initial assumption is incorrect. No such x exists meaning that $A \subseteq B \cup C$, as desired.

3) (15 pts) (PRF) Logic

Use the Laws of Logic and Rules of Inference to justify the following argument:

$u \wedge p$

$p \to \neg q$

$r \vee t \to q$

$s \vee t$

$\therefore s$

Please name the Law of Logic or Rule of Inference used in each step of your proof.

Solution.

1.	$u \wedge p$	Premise
2.	p	1, Conjunctive Simplification
3.	$p \to \neg q$	Premise
4.	$\neg q$	2, 3, Modus Ponens
5.	$r \vee t \to q$	Premise
6.	$\neg(r \vee t)$	4, 5, Modus Tollens
7.	$\neg r \wedge \neg t$	6, DeMorgan's Laws
8.	$\neg t$	7, Conjunctive Simplification
9.	$s \vee t$	Premise
10.	s	8, 9, Disjunctive Syllogism

Grading:

1 pt for the symbolic form of each step (10 pts total).
5 pts total for including the names of all the rules.

(If no names are given, -5 pts. If one or two of the names are missing or incorrect, -1 pt. If two to five are missing or incorrect, -2 pts. If more than five are missing or incorrect but the student tried to give some names, -3 pts.)

Note: There are other ways to reach the same conclusion, so make sure you give credit for other valid solutions.