Computer Science Foundation Exam

December 16, 2005

Section II A

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

Name: ______________________________

SSN: _______________________________

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work both of the following problems (1 and 2).**

1) Prove for all non-negative integers n that (2^{2n} - 3n - 1) is divisible by 9.

2) Prove or disprove the two assertions below about arbitrary sets A, B and C. Be sure to explain each step in your proof/disproof.

a) If $C \subseteq B$, then, $(A - B) \cup (B - C) = \neg C \cap (A \cup B)$.

b) If (A - B) - C = A - (B - C) then $A \cap C = \varnothing$.

Solution to Problem 1:

Prove for all non-negative integers n that (2^{2n} - 3n - 1) is divisible by 9.

Base case: n=0. Evaluating the given expression at n=0, we get $2^{2(0)}$ - 3(0) - 1 = 1 - 0 - 1 - 0. Since 0 is divisible by 9, the base case holds. (2 pts)

Inductive hypothesis: Assume for an arbitrary value of n=k that (2^{2k} - 3k - 1) is divisible by 9, namely that (2^{2k} - 3k - 1) can be expressed as 9c for some integer c. (2 pts)

Inductive step: Prove for n= k+1 that ($2^{2(k+1)}$ - 3(k+1) - 1) is divisible by 9, namely that ($2^{2(k+1)}$ - 3(k+1) - 1) can be expressed as 9c' for some integer c'. (3 pts)

$2^{2(k+1)}$ - 3(k+1) - 1 = 2^{2k+2} - 3k - 3 - 1 (2 pts)

= $2^2 2^{2k}$ - 3k - 4 (2 pts)

= 4(2^{2k}) - 3k - 4 (2 pt)

=4(2^{2k}) - 3k - 9k - 4 + 9k (4 pts)

= 4(2^{2k}) - 12k - 4 + 9k (2 pts)

= 4(2^{2k} - 3k - 1) + 9k (2 pt)

= 4(9c) + 9k, using the inductive hypothesis (2 pts)

= 9(4c+k), since c and k are integers, we have found a suitable integer c' = 4c+k that proves the assertion. (2 pts)

Solution to Problem 2:

Prove or disprove the two assertions below about arbitrary sets A, B and C. Be sure to explain each step in your proof/disproof.

a) If $C \subseteq B$, then, $(A - B) \cup (B - C) = \neg C \cap (A \cup B)$.

b) If (A - B) - C = A - (B - C) then A $\cap$ C = $\varnothing$.

a) $\neg C \cap (A \cup B) = (\neg C \cap A) \cup (\neg C \cap B)$**, using the distributive law**
= (A - C) $\cup$ (B - C), using the definition of set difference (2 pts)

Our job reduces to proving that $(A - B) \cup (B - C)$ = (A - C) $\cup$ (B - C). Since the component (B - C) is in both sets, we really just need to prove the following two assertions:

(1) $(A - B) \subseteq$ (A - C) $\cup$ (B - C). (1 pt)
(2) (A - C) $\subseteq (A - B) \cup (B - C)$

To prove (1), we must show that if x $\in (A - B)$, then x $\in$ (A - C) $\cup$ (B - C). Using the given information we have that x $\in$ A and x $\notin$ B. Since C $\subseteq$ *B, it follows that x*$\notin$ C. Thus, we can conclude that x $\in$ A - C, but definition of set difference and x $\in$ (A - C) $\cup$ (B - C), as desired. (5 pts for this proof)

To prove (2), we must show that if x $\in (A - C)$, then x $\in$ (A - B) $\cup$ (B - C). Using the given information we have that x $\in$ A and x $\notin$ C. From here, we have two cases: either x$\in$B or x$\notin$B. In the former case, x$\in$B-C and in the latter case x$\in$A-B. Thus, in all cases, we find that x $\in$ (A - B) $\cup$ (B - C) as desired. (5 pts for this proof)

Grading note: There are other ways to do this, adjust the grading criteria out of 13 points accordingly.

b) Use proof by contradiction. Assume the opposite, that A and C share a common element. Let one of these be x. Then we find that by definition of set difference x $\notin$ (A-B) - C, since C is subtracted out of another set. But, we also find that x$\notin$B - C for the same reason, but if this is the case and since x$\in$A, it follows that x$\in$ A - (B - C). This contradicts the given information that (A - B) - C = A - (B - C). It follows that the intersection of A and C must be empty.

Grading: 3 pts to set up the contradiction proof. 4 pts to show that x is not in the first set, 4 pts to show that x IS in the second set. 1 pt to conclude the proof.

Note: There are other ways to solve this also, adjust the criteria accordingly.

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Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ______________________________

SSN: ______________________________

In this section of the exam, there are four (4) problems.

You must do two (2) of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

PART B: Work any two of the following problems (3 through 6).

3) Relations

a) Let A = {1, 2, 3, 4, 5} and R be a binary relation over A such that R = { (1,1), (1, 3), (1,5), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (5, 1), (5, 3), (5,5)}. Is R reflexive, irreflexive, symmetric, anti-symmetric, or transitive? Justify each answer.

b) Let R and S be binary relations over Z. Prove or disprove: if R is transitive and S is transitive, then $R \circ S$ is also transitive.

a) R is not reflexive because (4,4) $\notin$ R.
R is not irreflexive because (1,1) $\in$ R.
R is not symmetric because (5,3) $\in$ R but (3,5) $\notin$ R.
R is not anti-symmetric because (1,3) $\in$ R and (3,1) $\in$ R.
R is not transitive because (4,2) $\in$ R, (2,4) $\in$ R, but (4,4) $\notin$ R.

Grading: 1 pt for saying each property is not upheld, and 2 pts for the proof.

b) This is false. Consider the following counter-example:

Let R = {(1,3), (2,4)}, S = {(3,2), (4,5)}. In this situation, both R and S are vacuously transitive. But $R \circ S$ = {(1,2), (2,5)}, which means that $R \circ S$ is NOT transitive because (1,2) $\in R \circ S$, (2,5) $\in R \circ S$, but (1,5) $\notin R \circ S$.

Grading: 4 points for completely specifying the counter-example and 6 points for explaining why it's a counter-example.

4) Functions

Let f and g be functions such that f: A→B, g: B→C, where A, B and C are finite sets.

a) Prove or disprove: if f is surjective and g is surjective, then g°f is surjective.

b) Prove or disprove: if f is surjective and g is injective, then g°f is injective.

a) We must prove the following:

For all elements c∈C, there exists an element a∈A such that g(f(a)) = c. (3 pts)

Consider an arbitrarily chosen element c. We know because g is surjective, that there must exists an element b∈B such that g(b) = c. (4 pts)

Furthermore, since f is surjective, for the b chosen above, we know there must exist an element a∈A such that f(a) = b. (4 pts)

It follows that g(b) = g(f(a)) = c, proving the assertion. (4 pts)

b) This is false. Consider the following counter-example:

Let A = {1,2,3}, B = {a,b} and C={x,y}.
Let f = {(1,a), (2,b), (3,b)} and g = {(a,x),(b,y)}.

f is surjective because B is covered, and g is injective because a and b map to different places, but here we have g°f = {(1,x),(2,y),(3,y)}, and g°f is NOT injective because g°f(2) = g°f(3).

Grading: 4 points to completely specify the counter-example and 6 points to explain why it's a counter-example.

5) Counting

a) Given sets A, B, and C such that $|A|+|B|+|C| = 30$, $|A \cap B \cap C| = 4$, and $|A \cup (B \cap C)| = 7$, find $|B \cup C|$. In order to receive full credit, you must apply valid counting rules instead of using a guess and check method with Venn Diagrams.

b) How many non-negative integer solutions are there to the equation $x + y + z + w = 13$ such that w does not exceed 5?

a) $|A \cup (B \cap C)| = |A| + |B \cap C| - |A \cap (B \cap C)|$, using the inclusion-exclusion principle on A and $B \cap C$. (4 pts)

$|A \cup (B \cap C)| = |A| + |B| + |C| - |B \cup C| - |A \cap (B \cap C)|$, using the inclusion exclusion principle on B and C. (5 pts)

Plugging in the given information we get:

$7 = 30 - |B \cup C| - 4$, so (2 pts)

$|B \cup C| = 19$. (1 pts)

b) Without the restriction on w, this is a straight combination with repetition problem. Using the standard formula, the number of non-negative integer solutions to the equation is $\binom{13+4-1}{4-1} = \binom{16}{3}$. From these solutions, we must subtract out the ones with w > 5. To do this, simply "assign" w=6, and revise the equation to be:

$x + y + z + w' = 7$, where w' is the "left-over" value above 6 for a given w in the original equation. There is a one-to-one correspondence between non-negative solutions to this equation and all solutions to the original equation with w > 5. The number of solutions to this adjusted equation is $\binom{7+4-1}{4-1} = \binom{10}{3}$. Subtracting these out, we get the final answer of $\binom{16}{3} - \binom{10}{3} = 440$.

Grading: 5 pts for $\binom{16}{3}$ and explanation, 5 pts for $\binom{10}{3}$ and explanation, 3 points for concluding the question.

Note: There are quite a few other ways to do this question. One other way involves setting w=0,1,2,3,4,5 and adding up the six possibilities. Grade accordingly.

6) Number Theory

a) Find the greatest common divisor of 682 and 187 using Euclid's Algorithm. Please show each step in the algorithm.

b) Find integers x and y such that 682x+187y = gcd(682,187), using the Extended Euclidean Algorithm. Please show each step in the algorithm.

a)

682 = 3x187 + 121 (2 pts)
187 = 1x121 + 66 (2 pts)
121 = 1x66 + 55 (2 pts)
66 = 1x55 + 11 (2 pts)
55 = 5x11, (1 pt), so gcd(682, 187) = 11 (1 pt)

b)

11 = 66 - 55, (1 pt)
55 = 121 - 66, so substituting we get (1 pt)

11 = 66 - (121 - 66) (2 pts)
11 = 2x66 - 121 (1 pt)
66 = 187 - 121, so substituting we get (1 pt)

11 = 2x(187-121) - 121 (2 pts)
11 = 2x187 - 3x121 (1 pt)
121 = 682 - 3x187, so substituting we get (1 pt)

11 = 2x187 - 3x(682-3x187) (2 pts)
11 = 2x187 - 3x682 + 8x187 (1 pt)
11 = 11x187 - 3x682, (1 pt)

so an ordered pair (x,y) that satisfies the given equation is (-3, 11). (1 pt)