**FOUNDATION EXAM (DISCRETE STRUCTURES)**

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

**PART A**: **Work both of the following problems (1 and 2).**

1. Let $Z = \{0, 1, -1, 2, -2, \ldots\}$ denote the set of all integers (zero, positive, and negative). Define a function $g: Z \to Z$ by the following formula:

$$g(m) = \begin{cases} 1 - m, \text{if } m \text{ is an even integer; otherwise,} \\ m + 3, \text{if } m \text{ is an odd integer.} \end{cases}$$

(Thus, for example, $g(0) = 1 - 0 = 1$; $g(1) = 1 + 3 = 4$; $g(-1) = -1 + 3 = 2$, etc.)
Prove that the function $g$ defines a bijection from $Z$ to $Z$; that is, prove that $g$ is an injection (one-to-one) and $g$ is a surjection (onto).

First, we prove the function $g$ is an injection (i.e., $g$ is one-to-one). We need to prove that if $g(m) = g(n)$ --- (1), where $m$ and $n$ are two integers, then $m = n$ --- (2).
Since $g(m)$ (and also $g(n)$) can be defined by two different formulas depending whether the argument $m$ (and $n$) is even or odd, we consider the following 3 cases:
(Case 1) Both $m$ and $n$ are even. In this case, (1) implies

$1 - m = 1 - n$; thus, $-m = -n$, and $m = n$.

So (2) is proved in this case.
(Case 2) Both $m$ and $n$ are odd. In this case, (1) implies

$m + 3 = n + 3$; thus, $m = n$ is true.

So (2) is proved in this case.
(Case 3) One of the two numbers is even, the other is odd. We may assume $m$ is even and $n$ is odd (that is, we name the even number $m$ and name the other $n$).
In this case, (1) implies (using the definition of the function $g$) that

$1 - m = n + 3$. Thus, $m + n = 1 - 3 = -2$.

However, since $m$ is even and $n$ is odd, $m + n$ must be odd, so it cannot be equal to $-2$. That is, we proved that when $m \neq n$ and $m$ is even and $n$ is odd, $g(m) = g(n)$ leads to a contradiction, which provided an indirect proof of (1) $\Rightarrow$ (2) in this case.
Therefore, we proved that (1) implies (2) in all cases, which proved the injection property of function $g$.

To prove $g: Z \to Z$ is a surjection (i.e. is onto), let $n \in Z$ denote an arbitrary integer from the co-domain $Z$. We need to prove there exists $m \in Z$ such that $g(m) = n$. We choose (solve for) $m$ depending on whether $n$ is even or $n$ is odd.
(Case 1) Suppose $n$ is even. In this case, choose $m = n - 3$. Note that $m$ is odd, this is because $n$ is even and $-3$ is odd, their sum is odd. Thus,

$g(m) = m + 3$, since $m$ is odd
$= (n - 3) + 3$, by substitution
$= n$.

(Case 1) Suppose $n$ is odd. In this case, choose $m = 1 - n$. Note that $m$ is even, this is because 1 is odd and $-n$ is odd, their sum must be even. Thus,

$g(m) = 1 - m$ , since $m$ is even
$= 1 - (1 - n)$, by substitution
$= n$.

Thus, we showed that for each $n \in Z$ there exists $m \in Z$ such that $g(m) = n$ in both cases, which proved that $g$ is a surjection.

2. Use induction on $n \geq 0$ to prove the following summation formula:

$$\sum_{i=0}^{n}(i+2)2^i = (n+1)2^{n+1}, \text{ for all } n \geq 0,$$

by completing the **induction hypothesis** and **the induction steps** (the Basis Step is given for your reference):

**Proof:**

(Basis Step) Consider $n = 0$. In this case,

The LHS $= \sum_{i=0}^{0}(i+2)2^i = (0+2)2^0 = 2$; and

the RHS $= (0+1)2^{0+1} = 2$.

Thus, LHS = RHS, so the Basis Step is proved.

(Induction Hypothesis) Consider $n = k$. We assume the following is true:

$$\sum_{i=0}^{k}(i+2)2^i = (k+1)2^{k+1}, \text{ for some } k \geq 0.$$

(Induction Step) Consider $n = k + 1$. We need to prove

$$\sum_{i=0}^{k+1}(i+2)2^i = (k+1+1)2^{k+1+1} = (k+2)2^{k+2} \text{ --- } (1).$$

The LHS of (1) $= \sum_{i=0}^{k}(i+2)2^i + (k+1+2)2^{k+1}$, by the definition of summation

$= (k+1)2^{k+1} + (k+3)2^{k+1}$, by the Induction Hypothesis

$= (k+1+k+3)2^{k+1}$, factoring

$= (2k+4)2^{k+1}$

$= 2(k+2)2^{k+1}$

$= (k+2)2^{k+2}$, because $2(2^{k+1}) = 2^{k+2}$

$=$ RHS of (1).

Thus, we proved the Induction Step.

By induction, we proved that the summation identity is true for all $n \geq 0$.

**PART B: Work any two of the following problems (3 through 6)**.

3. Let $A$ denote an arbitrary non-empty set, and let $R$ denote a binary relation, $R \subset A \times A$. Answer the following two parts **independently of each other**:

(a) Suppose $R$ is transitive. Prove that the inverse relation $R^{-1}$ is also transitive, where $R^{-1}$ is defined as $R^{-1} = \{(a, b) \mid (b, a) \in R\}$.

Let $(a, b) \in R^{-1}$ --- (1) and $(b, c) \in R^{-1}$ --- (2), we need to prove $(a, c) \in R^{-1}$ --- (3).

By the definition of $R^{-1}$, (1) implies $(b, a) \in R$ --- (4), and (2) implies $(c, b) \in R$ --- (5). Since $R$ is transitive by assumption, (4) and (5) together imply $(c, a) \in R$. Thus, $(a, c) \in R^{-1}$, by the definition of $R^{-1}$, which proves (3).

(b) Suppose $R \neq \varnothing$ and $R$ is irreflexive (that is, there does **not** exist any $a \in A$ such that $(a, a) \in R$). Prove that either $R$ is not symmetric or $R$ is not transitive.
We use the method of proof by contradiction. That is, we assume $R$ is symmetric and $R$ is transitive --- (1), then we show this leads to a contradiction.
Since $R \neq \varnothing$ by assumption, we let $(a, b) \in R$ --- (2) denote an arbitrary element of $R$. Thus, (2) implies $(b, a) \in R$ --- (3) because $R$ is symmetric as assumed in (1). Combining (2) and (3), we obtain $(a, a) \in R$ by using the transitive property of $R$ as assumed in (1). However, this means $R$ is not irreflexive, a contradiction to the assumption that $R$ is irreflexive.

4. Let $A$ denote an arbitrary non-empty set, and let $R$, $S$, and $T$ denote binary relations defined over $A$, i.e., $R \subset A \times A$, $S \subset A \times A$, and $T \subset A \times A$. Answer the following two questions **independently of each of other**:
(a) Prove $(R \circ (S \cap T)) \subset ((R \circ S) \cap (R \circ T))$.
Let $(a, b) \in R \circ (S \cap T)$ --- (1), we need to prove $(a, b) \in (R \circ S) \cap (R \circ T)$ --- (2).
Using the definition of relation composition "$\circ$", (1) implies there exists $c \in A$ such that $(a, c) \in R$ --- (3) and $(c, b) \in (S \cap T)$ --- (4).
Note that (4) implies $(c, b) \in S$ --- (5), and $(c, b) \in T$ --- (6), by the definition of $\cap$.
Combining (3) and (5), we obtain $(a, b) \in (R \circ S)$ by the definition of relation composition "$\circ$". Similarly, combining (3) and (6) yields $(a, b) \in (R \circ T)$.
Thus, $(a, b) \in (R \circ S) \cap (R \circ T)$, which proves (2).
(b) Suppose $A = \{a, b, c\}$. Use an example of relations $R$, $S$, and $T$ defined over this $A$ to show that $(R \circ (S \cap T)) \neq ((R \circ S) \cap (R \circ T))$.
Define the following relations over the set $A = \{a, b, c\}$:
$R = \{(a, a), (a, b)\}$; $S = \{(a, c)\}$; and $T = \{(b, c)\}$.
Thus, $S \cap T = \varnothing$, which implies $R \circ (S \cap T) = \varnothing$.
However, $R \circ S = \{(a, c)\}$ and $R \circ T = \{(a, c)\}$, so $(R \circ S) \cap (R \circ T) = \{(a, c)\}$.
Thus, this example shows $(R \circ (S \cap T)) \neq ((R \circ S) \cap (R \circ T))$.

5. Let A be a set that contains only the three strings $\lambda, x, y$ where x and y each have length 1 and $\lambda$ is the empty string.
(a) What is $|A^3|$?
(b) How many strings are there in $A^*$ that have length less than or equal to 5?
REMEMBER TO SHOW HOW YOUR ANSWERS ARE OBTAINED.

**Solution:**

There are 15 elements in $A^3$ given in the following:

(a) $A^3 = \{\lambda, x, y, xx, xy, yx, yy, xxx, xxy, xyx, xyy, yxx, yxy, yyx, yyy\}$. Thus, $|A^3| = 15$.
(b) There is 1 string of length 0.
plus 2 strings of length 1: x, y
plus 4 strings of length 2: xx, xy, yx, yy

plus 8 strings of length 3: xxx, xxy, xyx, …, yyy
plus 16 strings of length 4: xxxx, xxxy, …, yyyy
plus 32 strings of length 5: xxxxx, xxxxy, …, yyyyy
for a total of 63 strings of length less than or equal to 5.

6. You are given 12 playing cards that include 3 spades, 3 hearts, 3 diamonds, and 3 clubs (that is, 3 cards of each "suit.")
   (a) How many ways can one select 6 cards total from the 12 when choosing 1 spade, 1 heart, 2 diamonds, and 2 clubs?
   (b) How many ways can one select 6 cards total from the 12 when choosing at least 1 card from each suit?

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   **Solution:**

   (a) $\binom{3}{1}\binom{3}{1}\binom{3}{2}\binom{3}{2} = 3^4 = 81$.

   (b) The choices may be distributed among the suits either as 1 from one suit, 1 from another suit and 2 from each of the remaining two suits (1,1,2,2); or they may be distributed as 1,1,1,3 for a total of 6 cards where at least one comes from each suit. There are $\binom{4}{2} = 6$ ways to decide which two suits contribute 1 card in the 1,1,2,2 cases and there are $\binom{4}{1} = 4$ ways to choose which suit contributes 3 cards in the 1,1,1,3 cases. Thus, the answer is $6\binom{3}{1}\binom{3}{1}\binom{3}{2}\binom{3}{2} + 4\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{3} = 594$.