FOUNDATION EXAM (DISCRETE STRUCTURES)

Work each of the following problems and show the steps of your work carefully. The problem will be graded based on the completeness of the solution steps and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: Work both of the following problems.

1. Let A, B, and C be any three sets. Prove the following:
$(C - (A \cup B)) \cup (B \cap C) \cup (A \cap C) = C$.

(**Solution one**) We prove the set equality by applying set-theoretical laws as follows:
$(C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$
$= (C \cap \neg (A \cup B)) \cup (B \cap C) \cup (A \cap C)$, by the law $X - Y = X \cap \neg Y$
$= (C \cap \neg (A \cup B)) \cup ((B \cup A) \cap C)$, by the distributive law
$= (C \cap \neg (A \cup B)) \cup (C \cap (A \cup B))$, by the commutative laws for $\cup$ and for $\cap$
$= C \cap (\neg (A \cup B) \cup (A \cup B))$, by the distributive law
$= C \cap U$, by the law $\neg X \cup X = U$, where U stands for the universe
$= C$, by the law $X \cap U = X$.

(**Solution two**) We prove the set equality by proving the following two parts:
(1) Prove $(C - (A \cup B)) \cup (B \cap C) \cup (A \cap C) \subset C$; and
(2) Prove $C \subset (C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$. (**Note**: We use the notation $\subset$ to mean "subset of or equal to", which sometimes is denoted as $\subseteq$.)

To prove (1), let $x \in (C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$ ---- (3), we need to prove $x \in C$ ---- (4). From (3), we have three cases, by the definition of set union:
(Case 1) $x \in (C - (A \cup B))$. In this case, $x \in C$ and $x \notin (A \cup B)$. In particular, $x \in C$.
(Case 2) $x \in (B \cap C)$. In this case, $x \in B$ and $x \in C$. Thus, $x \in C$ is true.
(Case 2) $x \in (A \cap C)$. In this case, $x \in A$ and $x \in C$. Thus, $x \in C$ is true.
Therefore, $x \in C$ is true in all three cases, which proves (4).

To prove (2), let $x \in C$ ---- (5), we need to prove $x \in (C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$ ---- (6). Considering the element x relative to the set $(A \cup B)$, we have the following two cases, depending on whether $x \in (A \cup B)$ or not.
(Case 1) Suppose $x \in (A \cup B)$ is true. Thus, $x \in A$ ---- (7) or $x \in B$ ---- (8).
If (7) is true, since $x \in C$ from (5), so $x \in (A \cap C)$
$\subset (C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$ ---(9)
If (8) is true, since $x \in C$ from (5), so $x \in (B \cap C)$
$\subset (C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$ ---(10)
(Case 2) Suppose $x \in (A \cup B)$ is false, that is, $x \notin (A \cup B)$. Since $x \in C$ from (5), so $x \in (C - (A \cup B))$ ---- (11), by the definition of set difference. Since $(C - (A \cup B)) \subset (C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$ ---- (12), so (11) and (12) imply $x \in (C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$ --- (13).
Combining (9), (10), and (13), we proved that $x \in (C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$ in all cases, which proves (6).

2. Prove the induction step in an induction proof for the following summation identity:

$$\frac{1}{2^n} + \sum_{j=1}^{n} \frac{1}{2^j} = 1, \text{ where } n \text{ denotes an integer and } n \geq 1.$$

That is, state the induction hypothesis precisely, then prove the induction step based on the induction hypothesis.

(**Note**: The question doesn't ask for the Basis Step of the induction proof, but it is included in the following for a complete induction proof, for your reference.)
(Basis Step) Consider $n = 1$. In this case,

$$\text{The LHS} = \frac{1}{2} + \frac{1}{2} = 1 = \text{RHS, so the Basis Step is proved.}$$

(Induction Hypothesis) Consider $n = k$. Suppose

$$\frac{1}{2^k} + \sum_{j=1}^{k} \frac{1}{2^j} = 1, \text{ for some } k \geq 1.$$

(Induction) Consider $n = k + 1$. We need to prove

$$\frac{1}{2^{k+1}} + \sum_{j=1}^{k+1} \frac{1}{2^j} = 1 \text{ - - - - } (1).$$

$$\begin{aligned}
\text{Note that the LHS of } (1) &= \frac{1}{2^{k+1}} + \sum_{j=1}^{k} \frac{1}{2^j} + \frac{1}{2^{k+1}}, \text{ by the definition of summation} \\
&= \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \sum_{j=1}^{k} \frac{1}{2^j}, \text{ the commutative law} \\
&= \frac{2}{2^{k+1}} + \sum_{j=1}^{k} \frac{1}{2^j}, \text{ add fractions of a common denominator} \\
&= \frac{1}{2^k} + \sum_{j=1}^{k} \frac{1}{2^j}, \text{ cancel a common factor of 2} \\
&= 1, \text{ by the Induction Hypothesis.}
\end{aligned}$$

Thus, the Induction Step is proved.
By induction, we proved the summation identity of the question for all $n \geq 1$.

PART B: Work any two of the following problems:

3. (a) How many functions are there from a set with 3 elements to a set with 8 elements? (b) How many one-to-one functions are there from a set with 3 elements to a set with 8 elements? (c) How many onto functions are there from a set with 3 elements to a set with 8 elements? EXPLAIN YOUR ANSWER.

(a) Each element of the first set can be mapped to any of the 8 elements in the second set. Thus, the answer is 8^3. (b) The first element of the domain set can be mapped to any of 8 elements in the co-domain. The second element can be mapped to any of the remaining 7. The third can be mapped to any of the remaining 6. The answer is (8)(7)(6) or 336. (c) It is not possible to map 3 elements onto 8. (Think of trying to fill 8 mailboxes with 3 letters.) The answer is none.

4. Tom has 15 ping-pong balls each uniquely numbered with a number between 1 and 15. He also has a red box, a blue box, and a green box. (a) How many ways can Tom place the 15 distinct balls into the three boxes? (b) Suppose now that Tom has placed 5 ping-pong balls in each box. How many ways can he choose 5 balls from the three boxes so that he chooses at least one from each box? EXPLAIN YOUR ANSWER.

(a) Each solution is a function that maps 15 elements into 3. Thus, the answer is 3^{15}. (b) The 5 balls can be chosen either as 1,1,3 (1 from a box, 1 from another box, 3 from remaining box) or as 1,2,2. There are 3 ways to select as 1,1,3 (take the 3 balls from red or 3 from blue or 3 from green). There are 3 ways to select as 1,2,2. Thus, recalling that the balls are uniquely numbered, the answer is
3*C(5,1)*C(5,1)*C(5,3) + 3*C(5,1)*C(5,2)*C(5,2) = 2250.

5. Let A denote an arbitrary set, and let R denote a transitive relation over A, that is, $R \subset A \times A$, and for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$. Prove that the composition relation $R^2 = R \circ R$ is transitive.

   By the definition of the transitive property, to prove R^2 is transitive we need to prove the following: If $(x, y) \in R^2$ ---- (1) and $(y, z) \in R^2$ ---- (2), then $(x, z) \in R^2$ ---- (3). From (1) and the definition of R^2, there exists $p \in A$ such that $(x, p) \in R$ and $(p, y) \in R$ ---- (4). Since R is transitive by assumption, so (4) implies $(x, y) \in R$ ---- (5). Similarly, from (2) and the definition of R^2, there exists $q \in A$ such that $(y, q) \in R$ and $(q, z) \in R$ ---- (6). Since R is transitive by assumption, so (6) implies $(y, z) \in R$ ---(7). Combining (5) and (7), we have $(x, z) \in R^2$ by the definition of R^2; thus, (3) is proved.

6. Let p, q, r denote three propositions (i.e. statements). Prove that the logical expression (p and q)$\Rightarrow r$ is equivalent to the expression ($p \Rightarrow r$) or ($q \Rightarrow r$), but is not equivalent to the expression ($p \Rightarrow r$) and ($q \Rightarrow r$).

   (**Solution one**) We use logic rules and laws to prove the identity. (**Note**: The notation $\equiv$ means "equivalent to".)

   (p and q)$\Rightarrow r$ $\equiv \neg$ (p and q) or r, by the rule $x \Rightarrow y \equiv \neg x$ or y
   $\equiv$ ($\neg p$ or $\neg q$) or r, DeMorgan's law
   $\equiv$ ($\neg p$ or $\neg q$) or (r or r), idempotent law, i.e., x or $x \equiv x$
   $\equiv$ ($\neg p$ or r) or ($\neg q$ or r), associative law and commutative law
   $\equiv$ ($p \Rightarrow r$) or ($q \Rightarrow r$), by the rule $x \Rightarrow y \equiv \neg x$ or y

   To show that the expression (p and q)$\Rightarrow r$ is not equivalent to the expression ($p \Rightarrow r$) and ($q \Rightarrow r$), consider the case when p = T, q = r = F. Thus,

$(p \text{ and } q) \Rightarrow r$ = (T and F) $\Rightarrow$ F = F $\Rightarrow$ F = T. However,
$(p \Rightarrow r)$ and $(q \Rightarrow r)$ = (T $\Rightarrow$ F) and (F $\Rightarrow$ F) = F and T = F.

(**Solution two**) We use the truth table method to prove logical equivalence non-equivalence of the expressions.

p	q	r	p and q	$(p \text{ and } q) \Rightarrow r$	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r)$ or $(q \Rightarrow r)$	$(p \Rightarrow r)$ and $(q \Rightarrow r)$
F	F	F	F	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	T	F	F	T	T	F	T	F
F	T	T	F	T	T	T	T	T
T	F	F	F	T	F	T	T	F
T	F	T	F	T	T	T	T	T
T	T	F	T	F	F	F	F	F
T	T	T	T	T	T	T	T	T

Note that the column for the expression $(p \text{ and } q) \Rightarrow r$ (i.e., column 5) has the identical truth values as those in the column for the expression $(p \Rightarrow r)$ or $(q \Rightarrow r)$ (i.e. column 8), which proves the equivalence between the two expressions. However, the column for the expression $(p \Rightarrow r)$ and $(q \Rightarrow r)$ (i.e. the last column) is different, which proves the non-equivalence of this expression from the first two.