Computer Science Foundation Exam

August 12, 2005

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ________________________________

SSN: _________________________________

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work both of the following problems (1 and 2).**

1) (25 pts) Induction

Let *f(n)* be defined as follows:

$$f(1) = 1, \quad f(n) = \frac{(2^{n-1}-1)(f(n-1)+1)+2^{n-1}}{2^n-1}, \text{ for all } n > 1.$$

Prove, using induction on n, that $f(n) = \dfrac{2^{n+1}-n-2}{2^n-1}$ for all positive integers n.

2) Sets

a) Draw Venn diagrams representing the following Sets

i) (3 pts) $\overline{A \cap \overline{B}}$

ii) (3 pts) $\overline{A} \cap (B \cup C)$

b) (4 pts) Give an example of sets (A, B, and C) for which $A - C = B - C$, but $A \neq B$

c) Let A, B and C be any three sets. Prove or disprove the following propositions:

i) (5 pts) If $C = \emptyset$ then $(A - B) - C = A - (B - C)$

ii) (5 pts) If $A \subseteq B \cup C$, then either $A \subseteq B$ or $A \subseteq C$

iii) (5 pts) $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Solution to Problem 1:

(25 pts) Let f(n) be defined as follows:

$f(1) = 1$, $f(n) = \dfrac{(2^{n-1} - 1)(f(n-1) + 1) + 2^{n-1}}{2^n - 1}$, *for all n > 1.*

Prove, using induction on n, that $f(n) = \dfrac{2^{n+1} - n - 2}{2^n - 1}$ for all positive integers n.

Solution

Base case: n = 1, LHS = f(1) = 1, RHS = $\dfrac{2^{1+1} - 1 - 2}{2^1 - 1} = \dfrac{4 - 1 - 2}{2 - 1} = 1$

Thus, the equation is true for n=1. **(3 pts)**

Inductive hypothesis: Assume for an arbitrary positive integer n=k that

$$f(k) = \frac{2^{k+1} - k - 2}{2^k - 1} \text{ \textbf{(2 pts)}}$$

Inductive step: Prove that for n=k+1 that $f(k+1) = \dfrac{2^{k+2} - (k+1) - 2}{2^{k+1} - 1}$ **(3 pts)**

$$f(k+1) = \frac{(2^k - 1)(f(k) + 1) + 2^k}{2^{k+1} - 1} \text{ \textbf{(3 pts)}}$$

$$= \frac{(2^k - 1)(\frac{2^{k+1} - k - 2}{2^k - 1} + 1) + 2^k}{2^{k+1} - 1} \text{ \textbf{(3 pts)}}$$

$$= \frac{(2^{k+1} - k - 2 + 2^k - 1) + 2^k}{2^{k+1} - 1} \text{ \textbf{(4 pts)}}$$

$$= \frac{2^{k+1} + 2^k + 2^k - (k+1) - 2}{2^{k+1} - 1} \text{ \textbf{(2 pts)}}$$

$$= \frac{2^{k+1} + 2^{k+1} - (k+1) - 2}{2^{k+1} - 1} \text{ \textbf{(3 pts)}}$$

$$= \frac{2^{k+2} - (k+1) - 2}{2^{k+1} - 1} \text{ \textbf{(2 pts)}}$$

Solution to Problem 2:

a) Draw Venn diagrams representing the following Sets

i) (3 pts) $\overline{A \cap \overline{B}}$

ii) (3 pts) $\overline{A} \cap (B \cup C)$

b) (4 pts) Give an example of sets (A, B, and C) for which $A - C = B - C$, but $A \neq B$

c) Let A, B and C be any three sets. Prove or disprove the following propositions:

iv) (5 pts) If $C = \varnothing$ then $(A - B) - C = A - (B - C)$

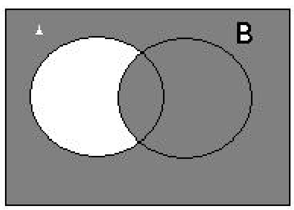
v) (5 pts) If $A \subseteq B \cup C$, then either $A \subseteq B$ or $A \subseteq C$

vi) (5 pts) $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

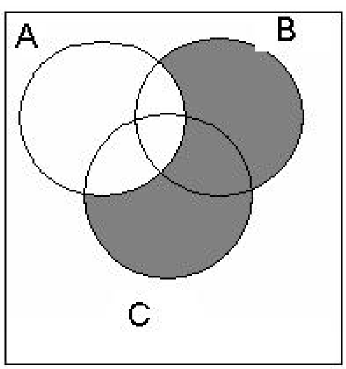
<u>Solution</u>

(a)

i) (3 pts)



ii) (3 pts)



(b) (4 pts) A ={1,2}, B= {1, 3}, C = {2, 3}

(c)

i) (5 pts)

Proof. Assume $C = \varnothing$. Let $x \in (A - B) - C$. It is trivial that $\forall S \forall R$, where S and R are sets,
$R = \varnothing$? $(S = S - R)$ is true. Thus $(A - B) - C = A - B$ (using $S = A - B$ as a substitution rule). Hence, $x \in A - B$. Also $B = B - C$, so when substituting this into the statement $x \in A - B$ we get $x \in A - (B - C)$. Thus $(A - B) - C \subseteq A - (B - C)$. Now let $x \in A - (B - \mathrm{C})$. Once again we know that $B - C = B$. From this we get $x \in A - B$. Making the substitution $A - B = (A - B) - C$ we now have $x \in (A - B) - C$. Thus $A - (B - C) \subseteq (A - B) - C$ Therefore if $C = \varnothing$ then $(A - B) - C = A - (B - C)$.

ii) (5 pts)

It can be **disproved** by the following counter example. Take $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 4\}$. Then $A \subseteq B \cup C = \{1, 2, 3, 4\}$, but A is neither a subset of B , nor a subset of C.

iii) (5 pts)

Proof. Take arbitrary $x \in A \cap (B \cup C)$ to show that $x \in (A \cap B) \cup C$. By distributive property $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, so we have $x \in (A \cap B) \cup (A \cap C)$. It means either $x \in (A \cap B)$ or $x \in (A \cap C)$, i.e. we have two cases to consider. Case 1: $x \in (A \cap B)$ implies $x \in (A \cap B) \cup C$ because $(A \cap B) \subseteq (A \cap B) \cup C$. Case 2: $x \in (A \cap C)$ implies $x \in C$ because $(A \cap C) \subseteq C$. From $x \in C$ we can imply $x \in (A \cap B) \cup C$ by $C \subseteq (A \cap B) \cup C$. So, for arbitrary $x \in A \cap (B \cup C)$ we can imply in any case that $x \in (A \cap B) \cup C$, that means $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Computer Science Foundation Exam

August 12, 2005

Section II B

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

Name: ______________________________

SSN: _______________________________

In this section of the exam, there are four (4) problems.

You must do two (2) of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

PART B: Work any two of the following problems (3 through 6).

3) Counting

Poker uses a standard deck of playing cards. A standard deck of playing cards includes 52 cards which are subdivided into four suits: spades, hearts, diamonds and clubs. There are 13 cards of each suit: Ace, two, three, four, five, six, seven, eight, night, ten, jack, queen and kick. A single hand in poker comprises five cards randomly chosen out of these 52 cards. Answer the following questions that deal (no pun intended =)) with poker:

(a) (4 pts) What are the total number of distinct poker hands that are possible? (Note: two hands are distinct if the first contains one card or more that is not in the second hand.)

(b) (9 pts) A hand with a single pair is a hand that contains two cards of the same kind of and three other cards of distinct kinds. (For example, a hand with 2 queens, a ten, a five and a three is a hand with a pair, but a hand with 2 queens, 2 fives and a seven would not be considered a hand with a single pair, and a hand with 3 queens, a five and a seven would not be considered a hand with a single pair, for the purposes of this question.) How many distinct poker hands have a single pair.

(c) (7 pts) A hand with two pairs contains one pair of cards of one kind, another pair of cards of a second kind, and a fifth card of a third, distinct kind. How many distinct poker hands with two pairs are there?

(d) (5 pts) A flush contains all five cards of the same suit. How many distinct poker hands have a flush?

<u>Solution</u>

Grading Note: Students did not have calculators at the exam. Please give full credit for any expression that is correct, even if it's not simplified to a number. Expressions that retain the underlying logic in answering the question should be preferred, but all correct responses with adequate explanation should receive full credit.

(a) (4 pts) A poker hand is simply a choice of 5 cards out of 52, thus, the possible number of distinct poker hands is $\binom{52}{5}$. **(4 pts, give partial if you deem necessary)**

(b) (9 pts) We first choose the kind for the pair. There are 13 ways to do this. **(1 pt)** Then we choose the 2 cards out of 4 possible, for the pair. There are $\binom{4}{2}$ ways to do this. **(2 pts)** We must choose 3 kinds of the remaining 12 available. This can be done in $\binom{12}{3}$ ways. **(3 pts)** Once we choose these kinds, we have 4 choices for each of these three cards. **(2 pts)** Multiplying each of these values together we obtain

$13\binom{4}{2}\binom{12}{3}4^3 = 1098240$ **(1 pt for multiplying)**

Note: A common mistake on this question will likely be counting permutations instead of combinations. For instance, we are choosing the last 3 kinds instead of permuting them. Take off 1 or 2 points in each instance this occurs.

(c) (7 pts) We first choose 2 kinds out of 13 for the 2 pairs. This can be done in $\binom{13}{2}$ ways. **(2 pts)** Next, for each pair, we choose 2 out of the 4 possible cards. For each pair this can be done in $\binom{4}{2}$ ways. **(2 pts)** Finally, we choose one remaining card from the possible 44 remaining. **(2 pts)** Multiplying each of these values together we obtain

$\binom{13}{2}\binom{4}{2}\binom{4}{2}44 = 123552$ **(1 pt for multiplying)**

(d) (5 pts) First we can choose one of four suits. **(2 pt)** Once that choice is made, we must then choose 5 cards out of 13 in that suit. **(2 pts)** Multiplying, we find a total of $4\binom{13}{5} = 5148$ possible flushes. **(1 pt for multiplying)**

4) Relations

(a) (10 pts) Let A = {1, 2, 3, 4, 5}. Let R be a binary relation over the set A. (Thus, $R \subseteq A \times A$.) In particular R = { (1, 2), (3, 1), (4, 2), (2, 4), (3, 3), (5, 2) }. Explain why R is not reflexive, irreflexive, symmetric, anti-symmetric or transitive.

(b) (15 pts) Let R be a binary relation over the set of integers defined as follows:

$R = \{ (a, b) \mid a - b \equiv 0 \bmod 10 \}$

Prove that R is an equivalence relation.

<u>Solution</u>

(a) (10 pts)

R is not reflexive because it does NOT contain (1,1).
R is not irreflexive because it DOES contain (3,3).
R is not symmetric because it contains (5,2) but does NOT contain (2,5).
R is not anti-symmetric because it contains both (4,2) and (2,4).
R is not transitive because it contains (5,2) and (2,4), but does NOT contain (5,4).

Grading: 2 points each, all or nothing. A specific counter-example is necessary, or description of the counter-examples in general is necessary.

(b) (15 pts)
We first prove that R is reflexive. Consider an arbitrary ordered pair (a,a). Since $a - a = 0$ and $0 \equiv 0 \bmod 10$, all ordered pairs of the form (a,a) are elements of R. **(4 pts)**

Next, we must show that R is symmetric. Namely, we must show that if $(a,b) \in R$, then $(b,a) \in R$. **(1 pt)** To prove this assume that $(a,b) \in R$. It follows that $a-b \equiv 0 \bmod 10$. **(1 pt)** We can multiply both sides of this equation by -1 to yield: $-1(a - b) \equiv (-1)0 \bmod 10$. **(1 pt)** Simplifying, we find that $-a + b \equiv 0 \bmod 10$, or $b - a \equiv 0 \bmod 10$. **(1 pt)** Thus, we can now conclude that $(b,a) \in R$. **(1 pt)**

Finally, we will show that R is transitive. We must show that if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$. **(1 pt)** We will assume that $(a,b) \in R$ and $(b,c) \in R$. Thus, we have:
$a - b \equiv 0 \bmod 10$
$b - c \equiv 0 \bmod 10$ **(1 pt)**

Add these two equations to yield:
$a - b + b - c \equiv 0 + 0 \bmod 10$ **(2 pts)**
$a - c \equiv 0 \bmod 10$ **(1 pt)**

Now, we can conclude that $(a,c) \in R$ as desired. (**1 pt)** Thus, R is an equivalence relationship.

5) Functions

(a) Determine which of the followings are functions with domain X.

i) (3 pts) X = {1, 3, 5, 7, 8} and R ={(1,7), (3,5), (5,3), (7, 7), (8,5)}

ii) (3 pts) X = {-2, -1, 0, 1} and R = {(-2, 6), (0, 3), (1, -1)}

iii) (3 pts) X is the set of real numbers and, for x ∈ X,
$g(x) = x^2 - 3x + 2$, assume that the codomain is also X

iv) (3 pts) X is the set of real numbers and, for x ∈ X,
$g(x) = \sqrt{x^2 - 3x + 2}$, assume that the codomain is also X

v) (3 pts) X is the set of real numbers and, for x ∈ X, $g(x) = \log_2 x$, , assume that the codomain is also X

(b) Let Z = {...−2, −1, 0, 1, 2, ...} denote the set of integers. Suppose f : Z→Z is a function, defined by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases}$$

i) (5pts) Prove or disprove that f is one-to-one (injective)

ii) (5pts) Prove or disprove that f is onto (surjective).

<u>**Solution**</u>

(a)

i) (3 pts) function

ii) (3 pts) not a function

iii) (3 pts) function

iv) (3 pts) not a function

v) (3 pts) not a function

(b)

i) (5pts) It can be **disproved** that f is one-to-one by the following counter-example. We have that $f(4)=f(1)=2$.

ii) (5pts) **Proof.** To prove that function f is onto, take arbitrary integer $k \in$ Z, to show that we can always find the pre-image of k under f , i.e. an integer n such that $f(n) = k$. Namely, take $n = 2k$, then, $f(2k) = k$, because $2k$ is even.

6) Number Theory

(a) (12 pts) Find the greatest common divisor of 585 and 432.

(b) (8 pts) Using your answer from part a, find the least common multiple of 585 and 432.

(c) (5 pts) What are all of the integer solutions for x and y to the equation 15x + 33y = 214?

Solution

(a) (12 pts)
585 = 1x432 + 153 **grading: 2 pts for each step, 12 points total**
432 = 2x153 + 126
153 = 1x126 + 27
126 = 4x27 + 18
27 = 1x18 + 9
18 = 2x9
GCD(585, 432) = 9.

(b) (8 pts) Using the fundamental theorem of arithmetic, it can be proven that gcd(a,b)xlcm(a,b) = ab. Using this equation, we find that:

lcm(a,b) = ab/gcd(a.b). **(5 pts for stating this)**

Applying this formula to this question, we have:

lcm(585, 432) = 585x432/9 = 585x48 = 28080 **(3 pts for the arithmetic)**

(c) (5 pts) We can rewrite the equation as follows:

3(5x+11y) = 214. **(3 pts for rewriting the equation like this)**

Since 214 is NOT divisible by 3 and 5d+11y must be integral since both x and y must be, there are NO integer solutions to the equation above. **(2 pts for this conclusion)**