Computer Science Foundation Exam

August 6, 2004

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ________________________________

SSN: _________________________________

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the total exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone

Credit cannot be given when your results are

unreadable.

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer both problems from Part A and two problems (out of four choices) from Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work both of the following problems (1 and 2).**

1) (25 pts) Let T_n denote the n^{th} Triangle number. In particular, $T_n = \sum_{i=1}^{n} i$, for all positive integers n. Using induction on n, prove that the following summation holds for all positive integers n:

$$\sum_{i=1}^{n} \frac{1}{T_i} = 2\left(1 - \frac{1}{n+1}\right)$$

(Hint: You should solve for T_n in a closed form solution in terms of n before working on this problem.)

2) a) (10 pts) Show that for arbitrary sets, A, B and C, taken from the universe {1, 2, 3, 4, 5} that the following two claims are not always true by using a single counter-example for each:

i) If $A \cap B \subseteq C$, then $C \subseteq A \cup B$.
ii) If $C \subseteq A \cup B$, then $A \cap B \subseteq C$.

b) (15 pts) Use the laws of logic to prove the following expression is a tautology: (Note: You must list each rule you use. The only rules you can combine together in one step are the commutative and associative rules to reorder and group terms more quickly. Hint: First try to "distribute" the not signs before you simplify any other part of the expression.)

$$((p \vee r) \vee ((q \wedge r) \vee p)) \vee \neg ((\neg(p \wedge q)) \wedge p)$$

Solution to Problem 1:

Let T_n denote the n^{th} Triangle number. In particular, $T_n = \sum_{i=1}^{n} i$, for all positive integers n. Using induction on n, prove that the following summation holds for all positive integers n:

$$\sum_{i=1}^{n} \frac{1}{T_i} = 2\left(1 - \frac{1}{n+1}\right)$$

(Hint: You should solve for T_n in a closed form solution in terms of n before working on this problem.)

<u>Solution</u>

First, note that $T_n = \frac{n(n+1)}{2}$, using the formula for the sum of the first n positive integers.**(2 pts)** We will now prove using induction on n, that for all positive integers n, $\sum_{i=1}^{n} \frac{1}{T_i} = 2\left(1 - \frac{1}{n+1}\right)$.

Base case n=1: LHS = $\sum_{i=1}^{1} \frac{1}{T_i} = \frac{1}{T_1} = \frac{1}{1} = 1$ RHS = $2\left(1 - \frac{1}{1+1}\right) = 2\left(\frac{1}{2}\right) = 1$ **(2 pts)**

Thus, the formula is true for n=1.

Inductive hypothesis: Assume for an arbitrary positive integer value of n=k that

$$\sum_{i=1}^{k} \frac{1}{T_i} = 2\left(1 - \frac{1}{k+1}\right). \text{ \textbf{(2 pts)}}$$

Inductive step: Prove for n=k+1 that $\sum_{i=1}^{k+1} \frac{1}{T_i} = 2\left(1 - \frac{1}{(k+1)+1}\right) = 2\left(1 - \frac{1}{k+2}\right)$

(2 pts)

$\sum_{i=1}^{k+1} \frac{1}{T_i} = \sum_{i=1}^{k} \frac{1}{T_i} + \frac{1}{T_{k+1}}$ **(3 pts)**

$= 2\left(1 - \frac{1}{k+1}\right) + \frac{1}{T_{k+1}}$, using the inductive hypothesis. **(3 pts)**

$$= 2\left(1 - \frac{1}{k+1}\right) + \frac{1}{\frac{(k+1)(k+2)}{2}}$$, using the closed form of T_n. **(2 pts)**

$$= 2\left(1 - \frac{1}{k+1}\right) + \frac{2}{(k+1)(k+2)}$$ **(2 pts)**

$$= 2\left[1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}\right]$$ **(1 pt)**

$$= 2\left[1 - \frac{(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}\right]$$ **(2 pts)**

$$= 2\left[1 - \frac{(k+2)-1}{(k+1)(k+2)}\right]$$ **(2 pt)**

$$= 2\left[1 - \frac{(k+1)}{(k+1)(k+2)}\right]$$ **(1 pt)**

$$= 2\left[1 - \frac{1}{(k+2)}\right]$$, proving the inductive step. **(1 pts)**

Solution to Problem 2:

a) (10 pts) Show that for arbitrary sets, A, B and C, taken from the universe {1, 2, 3, 4, 5} that the following two claims are not always true by using a single counter-example for each:

i) If $A \cap B \subseteq C$, then $C \subseteq A \cup B$.
ii) If $C \subseteq A \cup B$, then $A \cap B \subseteq C$.

Solution

i) Let A = {1}, B = {2}, C = {3}. $A \cap B = \emptyset$ so $A \cap B \subseteq C$ is automatically true for this example, but clearly $3 \in C$ while $3 \notin A \cup B$, thus, $C \subseteq A \cup B$ is false for this example. **(5 pts - 2 pts for stating the elements in A, B and C, 1 point for showing that the if part is true, 2 pts for showing that the then part is false.)**

ii) Let A = {1, 2}, B = {1, 2} and C = {1}. In this example, $C \subseteq A \cup B$ is true since $1 \in$ A. But, $A \cap B$ = {1, 2} in this example, and since $2 \in A \cap B$, but $2 \notin C$, it follows that $A \cap B \subseteq C$ is false for this example. **(5 pts - 2 pts for stating the elements in A, B and C, 1 point for showing that the if part is true, 2 pts for showing that the then part is false.)**

Solution to Problem 2 con't:

b) (15 pts) Use the laws of logic to prove the following expression is a tautology: (Note: You must list each rule you use. The only rules you can combine together in one step are the commutative and associative rules to reorder and group terms more quickly. Hint: First try to "distribute" the not signs before you simplify any other part of the expression.)

$((p \vee r) \vee ((q \wedge r) \vee p)) \vee \neg ((\neg(p \wedge q)) \wedge p)$

Solution

$((p \vee r) \vee ((q \wedge r) \vee p)) \vee \neg ((\neg(p \wedge q)) \wedge p) \Leftrightarrow$

$((p \vee r) \vee ((q \wedge r) \vee p)) \vee \neg\neg(p \wedge q) \vee \neg p$ $\Leftrightarrow$ (De Morgans) **3 pts**

$((p \vee r) \vee ((q \wedge r) \vee p)) \vee (p \wedge q) \vee \neg p$ $\Leftrightarrow$ (Double Negation) **3 pts**

$(\neg p \vee p) \vee ((p \vee r) \vee (q \wedge r) \vee (p \wedge q))$ $\Leftrightarrow$ (Comm+Assoc, or) **3 pts**

$T \vee ((p \vee r) \vee (q \wedge r) \vee (p \wedge q))$ $\Leftrightarrow$ (Inverse Law) **3 pts**

T $\Leftrightarrow$ (Domination Law) **3 pts**

Note: There are longer ways to do this question. If someone takes a longer way, only deduct 1 or 2 points per error.

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Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ______________________________

SSN: ______________________________

In this section of the exam, there are four (4) problems.

You must do two (2) of them.

Each counts for 25% of the total exam grade.

You must clearly identify the problems you are solving.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone

Credit cannot be given when your results are

unreadable.

PART B: Work any two of the following problems (3 through 6).

3) Define the edit distance between two strings *a* and *b* of equal length to be the minimum number of letter substitutions you must make in string *a* in order to obtain string *b*. For example, the edit distance between the strings "HELLO" and "JELLO" is 1, since only 'J' must be substituted for 'H' in order to obtain the second word from the first. Also, the edit distance between "HELLO" and "JELLY" is two, since in addition to the first substitution described, a 'Y' must be substituted for 'O'. (Note: For all three parts of this question, assume that all strings are *case insensitive*.)

a) (10 pts) How many alphabetic strings of length 5 have an edit distance of 1 from the string "HELLO"?

Solution

You can choose one of the five locations in the string to make a change. **(3 pts)** For each of these five choices, we can change the letter to 25 other options. (It's not 26 since we can't change the letter to itself.) **(4 pts)** Thus, using the multiplication principle, there are 5x25 = 125 total strings with an edit distance of 1 from "HELLO". **(3 pts)**

b) (5 pts) Let n be your answer to question a. One may argue that the number of alphabetic strings of length 5 that have an edit distance of 2 from the string "HELLO" is n^2. In essence, one would argue that in order to find a string with an edit distance of 2 away from "HELLO", one must change one letter at random, and then repeat that operation on the intermediate string. (i.e. "HELLO" $\rightarrow$ "JELLO" $\rightarrow$ "JELLY") To count how many ways this can be done, since the first operation is independent of the second, we would simply use the multiplication principle and multiply the number of ways the first operation can be done by the number of ways the second operation can be done. Both of these values are n, leading to a final answer of n^2. What is the flaw with this argument?

Solution

Not all distinct ordered pairs of operations lead to distinct strings. Consider the two following distinct ordered pairs of operations:

"HELLO" $\rightarrow$ "JELLO" $\rightarrow$ "JELLY" and
"HELLO" $\rightarrow$ "HELLY" $\rightarrow$"JELLY"

In the n^2 count, both of these two operations would be counted for two different words with an edit distance of 2 from "HELLO". But, as we can see, they should really only be counted as one word. **(5 pts)**

One may actually say then, that we can simply divide n^2 by 2 to obtain our answer. But this also, is faulty. This doesn't take into account, the following type of ordered pair of operations:

"HELLO" $\rightarrow$ "JELLO" $\rightarrow$ "HELLO" or
"HELLO" $\rightarrow$ "JELLO" $\rightarrow$ "MELLO"

In spite of the fact that both operations are distinct, they don't result in a final string that is actually an edit distance of 2 from "HELLO"

(Note: For grading purposes, finding any single flaw with the given argument should deserve full credit.)

c) (10 pts) Determine the actual number of alphabetic strings of length 5 with an edit distance of 2 from the string "HELLO".

<u>Solution</u>

Out of the 5 characters, we must choose exactly 2 to edit. This can be done in $\binom{5}{2} = 10$ ways, since we are choosing 2 characters out of 5. **(4 pts)** For each of the two characters we change, we have exactly 25 possible choices. The choice of one character is completely independent of the other, so, we can change the characters in 25x25 = 625 ways.**(3 pts)** Using the multiplication principle, multiply the choices of pairs of characters to change with the number of ways to change them to obtain 10x625 = 6250 as the final answer. **(3 pts)**

4) Given three arbitrary sets A, B, C and functions f:A→B and g:B→C, answer both parts a and b below.

a) Fill in each of the blank cells in the table below by using one of the following phrases that best fits in it: "invertible, non-invertible, cannot be determined".

Solution

Case #	**f**	**g**	**g°f**
1	invertible	invertible	***invertible***
2	invertible	non-invertible	***non-invertible***
3	non-invertible	invertible	***non-invertible***
4	non-invertible	non-invertible	***cannot be determined***
5	onto	***invertible***	invertible

(Grading: 2 pts each, no partial credit.)

b) Provide a formal proof of one of your answers by first stating clearly what you want to prove. You may refer to the theorems given below.

Hints: g°f is the composition of f and g, which is a function from A to C. Given below are some theorems you may find useful:

Theorem 1: Let f:A→B and g:B→C. If g°f is onto, then g is onto.
Theorem 2: Let f:A→B. f is invertible if and only if f is one-to-one and onto.
Theorem 3: Let f:A→B and g:B→C. If g°f is one-to-one, then f is one-to-one.

Solution

Case 1: Both f and are invertible, show g°f is invertible.

To show g°f is invertible, we will show it is one-to-one and onto by Theorem 2.

Show g°f is one-to-one: g°f(a1)=g°f(a2) ⇒ g(f(a1))=g(f(a2)) but g is one-to-one, thus, f(a1)=f(a2) but f is also one-to-one, thus a1=a2. Thus g°f is one-to-one.

Show g°f is onto: For any c∈C, there exists b∈B such that g(b)=c because g is onto. But since f is onto, there exists a∈A such that f(a)=b. Substituting it into the previous equation, we get: for any c∈C, there exists a∈A such that g(f(a))=c. Thus g°f is onto. Thus, g°f is invertible.

(Grading: 7 pts for showing g°f is one-to-one and 8 pts for showing g°f is onto. In each case, 2 pts for stating what needs to be proved, and 2 or 3 points for each step, so that the totals work out.)

Case 2: f is invertible g is not invertible, show g°f is not invertible.

Assume g°f is invertible. By Theorem 2, it is onto, and by Theorem 1, g is onto. Thus by Thereom 2 again, g is not one-to-one. Let b1 and b2 be two distinct members of B such that g(b1)=g(b2)=c∈C. Since f is one-to-one and onto, there has to be two distinct members a1 and a2 of A such that f(a1)=b1 and f(a2)=b2. Therefore g°f(a1)= c= g°f(a2). But that means g°f is not one-to-one. Contradiction. Thus, g°f is not invertible.

(Grading: 3 pts for stating what to prove. 3 pts for each step that follows.)

Case 3: f is not invertible g is invertible, show g°f is not invertible.

Use proof by contradiction. Assume g°f is invertible. Similar to Case 2, we will have f is not onto, this time by Theorem 3. Let b∈B such that there is no a∈A such that f(a)=b (because f is not onto such a and b will exist). Let g(b)=c ∈ C (because g is a function such a c will exist). Since g°f is onto (from A to C by Thereom 2), there has to be a member of A, x, such that g°f(x)=c. Thus, g(f(x))=c. But g(b)=c. And g is one-to-one, Therefore f(x)=b. Contradiction. Thus, g°f is not invertible.

(Grading: 3 pts for stating what to prove. 3 pts for each step that follows.)

Case 4: neither f nor g is invertible, show g°f may or may not be invertible.

Consider A=C={1}, B={a,b}, and f={(1,a)}, g={(a,1),(b,1)}. Function f is not invertible because it is not onto. Function g is not invertible because it is not one-to-one. But, g°f={(1,1)} is invertible.

Also consider A={1}, B={a,b}, C={1,2,3} and f={(1,a)}, g={(a,1),(b,1)}. Function f is not invertible because it is not onto. Function g is not invertible because it is not one-to-one or onto. g°f={(1,1)} is **NOT** invertible because it is not onto.

(Grading: 9 pts to specify one case, 6 pts to specify the other case. Each detail for each case is worth 1 or 2 pts. with the disgression given to the grader.)

Case 5: f is onto, g°f is invertible, show g is invertible.

Use proof by contradiction. Assume g is not invertible. Since g°f is invertible, then it is one-to-one by Theorem 2, and thus, f is invertible by Theorem 2 again. By case 2, we can conclude that g°f is not invertible, which is a contradiction. Thus g is invertible.

Grading: 4 pts to set up the proof, 4 pts to apply Theorem 2 each time, 3 points to arrive at the contradiction.

For all of these, grade accordingly if students solve the question differently.

5) Given an arbitrary set A, relation R on A, i.e. $R \subseteq A \times A$, satisfies the following property P: $\forall a \in A\ \forall b \in A,\ |R \cap \{(a,b), (b,a)\}| \leq 1$.

Answer the following questions and justify each one of your answers by a simple proof or a (counter-)example.

(i) Can R be symmetric?
(ii) Is R necessarily anti-symmetric?
(iii) Is R necessarily transitive?
(iv) Can R be reflexive?

<u>Solution</u>

(i) Yes. We could give an example, as this is an existence proof. Let $A=\{1\}$ and $R=\{(1,1)\}$. R is symmetric. **(6 pts)**

(ii) Yes. If $(a, b), (b, a) \in R$, then $a=b$. **(6 pts)**

(iii) No. Let $A=\{1,2,3\}$ and $R=\{(1, 2), (2,3)\}$. Clearly, (1,3) does not have to belong to R. **(6 pts)**

(iv) Yes. For b equals a in the property P for all a in A, $R \cap \{(a,a), (a,a)\}$ <u>can</u> have cardinality one, which has to be (a,a). Alternatively, we could give an example, as this is an existence proof. **(6 pts)**

Award 1 bonus point for getting each part correct.

6) Prove that if $n \in Z^+$ and n>1 and there exists no prime number $p \le \sqrt{n}$ such that p | n, then n is a prime number.

Hints:

Every positive integer, except 1, either prime or composite.
Prime numbers are 2, 3, 5, 7, 11…
Every composite natural number (4,6,8,9,10,…) has a prime divisor.

<u>Solution</u>

Assume otherwise, n is not prime. Then, n is composite by the first hint. Then n=n1*n2 for some n1 and n2 such that $1<n1<n$ and $1<n2<n$ by definition of composite numbers. Thus, either n1 or n2 is less than or equal to $\sqrt{n}$ (for otherwise their multiplication would be greater than n). Without loss of generality, lets assume $n1 \le \sqrt{n}$. If n1 is a prime number than this is a contradiction. If n1 is a composite number, then by the last hint, n1 has a prime divisor p (for which $p<n1$, therefore n has a prime divisor $p<\sqrt{n}$, which is a contradiction. Therefore, n is prime.

Grading: 4 pts for setting up proof by contradiction.
4 pts for expressing n=pq, where p and q are integers greater than 1.
10 pts for showing that if $p < q$, then $p < \sqrt{n}$
7 pts for arguing that either p is prime or that p contains a prime factor less than itself, which contradicts the given information.