

Computer Science Foundation Exam

August 2, 2002

Section II A

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

Name: _____

SSN: _____

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the total exam grade.

Show the steps of your work carefully.

**Problems will be graded based on the completeness of
the solution steps and not graded based on the answer alone**

**Credit cannot be given when your results are
unreadable.**

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: Work both of the following problems (1 and 2).

1) Whenever Binary Billy acts up, his punishment is to write binary numbers on the board. He always starts writing 0, 1, 10, 11, 100, etc. Depending on the severity of behavior, Billy has to write all the binary numbers starting at 0 upto all binary numbers with a certain number of digits. For example, if Billy's bad behavior was rated at a 5, then Billy would have to write all the binary numbers from 0 through 11111. Let $B(n)$ denote the total number of binary *digits* Billy must write for a bad behavior rating of n . Using induction on n , prove that $B(n) = (n-1)2^n + 2$, for all positive integers n .

2) Prove the following assertions for sets A and B from an universe U without using Venn Diagrams or membership tables:

a.) $A \subseteq B$ if and only if $A \cap \neg B = \emptyset$.

b.) $A \subseteq B$ if and only if $\neg A \cup B = U$.

Solution to Problem 1:

Whenever Binary Billy acts up, his punishment is to write binary numbers on the board. He always starts writing 0, 1, 10, 11, 100, etc. Depending on the severity of behavior, Billy has to write all the binary numbers starting at 0 upto all binary numbers with a certain number of digits. For example, if Billy's bad behavior was rated at a 5, then Billy would have to write all the binary numbers from 0 through 11111. Let $B(n)$ denote the total number of binary *digits* Billy must write for a bad behavior rating of n . Using induction on n , prove that $B(n) = (n-1)2^n + 2$, for all positive integers n .

Base case. $n=1$. Billy must write two digits, 0 and 1, thus $B(1) = 2$. Looking at the right hand side we find it equal to $(1-1)2^1 + 2 = 2$, thus the given formula is true for $n=1$.

Inductive hypothesis: Assume that for an arbitrary value of $n=k$ that $B(k) = (k-1)2^k + 2$.

Inductive step: Prove, under the inductive hypothesis, that for $n=k+1$, $B(k+1) = ((k+1)-1)2^{k+1} + 2 = k2^{k+1} + 2$.

Let $b(n)$ = number of binary numbers with EXACTLY n digits. For example, $b(2) = 2$ since there are two binary numbers, 10 and 11 with exactly 2 digits.

In particular, using the multiplication principle, we have that $b(n) = 2^{n-1}$. This is because the first binary digit of the n digits must be 1. The rest can either be 0 or 1. Thus, we have 2 choices for each of the $n-1$ remaining digits. Since each of these choices are independent, we multiply 2 by itself $n-1$ times to obtain the total number of possible binary numbers with exactly n digits.

Now, to prove the assertion:

$$\begin{aligned} B(k+1) &= B(k) + (k+1)*b(k+1), \text{ since we are adding } b(k+1) \text{ numbers with } k+1 \text{ digits} \\ &= (k-1)2^k + 2 + (k+1)2^{(k+1)-1} \\ &= (k-1)2^k + 2 + (k+1)2^k \\ &= ((k-1)+(k+1))2^k + 2 \\ &= (2k)2^k + 2 \\ &= k2^{k+1} + 2 \end{aligned}$$

Prove the following assertions for sets A and B from an universe U without using Venn Diagrams or membership tables:

(Note: when I say $A \not\subseteq B$, I mean A is not a *subset*, not A is not a *proper subset*.)

a.) $A \subseteq B$ if and only if $A \cap \neg B = \emptyset$.

First we will show $A \not\subseteq B \supset A \cap \neg B = \emptyset$.

In order to show that the intersection of two sets is empty, we must show that the two sets, in this case A and $\neg B$ share no common elements. Assume the contrary, that there exists an element x such that $x \in A$ and $x \in \neg B$. By definition of a complement set, then $x \notin B$. But, if $x \in A$ and $x \notin B$, we can deduce that $A \not\subseteq B$. But, this contradicts our given information. Thus, our initial assumption, that $A \cap \neg B$ was non-empty is incorrect and we have shown that $A \cap \neg B = \emptyset$.

Now, we must prove $A \cap \neg B = \emptyset \supset A \subseteq B$.

Let's prove the contrapositive of this statement:

$A \not\subseteq B \supset A \cap \neg B \neq \emptyset$.

If $A \not\subseteq B$, then there exists an element x such that $x \in A$ and $x \notin B$. But if $x \notin B$, then we know that $x \in \neg B$. It follows that $x \in A \cap \neg B$. Thus, $A \cap \neg B \neq \emptyset$, as desired.

b.) $A \subseteq B$ if and only if $\neg A \cup B = U$.

First let's show that $A \subseteq B \supset \neg A \cup B = U$.

In order to show that $\neg A \cup B = U$, we must show that there is no element x such that $x \notin \neg A \cup B$. Assume such an x exists. Then we have that $x \notin \neg A \cup B$. If x is not an element of this union, then we can deduce that $x \notin \neg A$ and $x \notin B$. If this is the case, then $x \in A$. But that would contradict the given information that $A \subseteq B$. Thus, no such element x exists and we can conclude that $\neg A \cup B = U$.

Now, let's show that $\neg A \cup B = U \supset A \subseteq B$.

We can prove the contrapositive of this statement:

$A \not\subseteq B \supset \neg A \cup B \neq U$.

If $A \not\subseteq B$, then there exists an element $x \in U$ such that $x \in A$ and $x \notin B$. But if this is the case, it is clear that $x \notin \neg A$ and $x \notin B$, so $x \notin \neg A \cup B$. It follows that $\neg A \cup B \neq U$.

Computer Science Foundation Exam
August 3, 2002

Section II B

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

Name: _____

SSN: _____

In this section of the exam, there are two (4) problems.

You must do two (2) of them.

Each counts for 25% of the total exam grade.

You must clearly identify the problems you are solving.

Show the steps of your work carefully.

**Problems will be graded based on the completeness of
the solution steps and not graded based on the answer alone**

Credit cannot be given when your results are

unreadable.

3)

a) Show that the logical expression $((p \Rightarrow q) \Rightarrow (q \Rightarrow p))$ is equivalent to the logical expression $p \vee \neg q$, without the use of a truth table.

$$((p \supset q) \supset (q \supset p)) \hat{=} \emptyset(p \supset q) \dot{\cup} (q \supset p), \text{ using the definition of an implication}$$

$$\hat{=} (\emptyset(\emptyset p \dot{\cup} q)) \dot{\cup} (\emptyset q \dot{\cup} p), \text{ using the definition of an implication}$$

$$\hat{=} (p \dot{\cup} \emptyset q) \dot{\cup} (\emptyset q \dot{\cup} p), \text{ using De Morgan's law, double negation}$$

$$\hat{=} (p \dot{\cup} (p \dot{\cup} \emptyset q)) \dot{\cup} \emptyset q, \text{ using associative/commutative of or}$$

$$\hat{=} p \dot{\cup} \emptyset q, \text{ using the absorption law.}$$

b) The boolean function $f(p, q, r)$ is represented below in a truth table.

p	q	r	f(p,q,r)
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	F
T	F	F	F
T	F	T	F
T	T	F	T
T	T	T	F

Write out an expression for $f(p, q, r)$ in terms of $p, q,$ and $r,$ only using the boolean operators \wedge and \neg . (Hints: A logical expression remains unchanged if you negate it twice. Use DeMorgan's Law.)

$$\begin{aligned} f(p, q, r) &= (\emptyset p \dot{\cup} \emptyset q \dot{\cup} r) \dot{\cup} (p \dot{\cup} q \dot{\cup} \emptyset r) \\ &= \emptyset \emptyset ((\emptyset p \dot{\cup} \emptyset q \dot{\cup} r) \dot{\cup} (p \dot{\cup} q \dot{\cup} \emptyset r)) \\ &= \emptyset (\emptyset (\emptyset p \dot{\cup} \emptyset q \dot{\cup} r) \dot{\cup} \emptyset (p \dot{\cup} q \dot{\cup} \emptyset r)) \end{aligned}$$

4) Intelligent life has finally been discovered on Mars! Upon further study, linguistic researchers have determined several rules to which the Martian language adheres:

- 1) The Martian alphabet is composed of three symbols: a, b, and c.
- 2) Each word in the language is a concatenation of four of these symbols.
- 3) Each command in the language is composed of three words.

a.) How many distinct commands can be created if words in a single command can be repeated and two commands are identical only if the three words AND the order in which the words appear are identical? (Thus, the commands "aaca baaa aaca" and "baaa aaca aaca" are two DIFFERENT commands.)

Total of 12 symbols in a command. For each of these symbols, we have 3 choices without any restrictions. These choices are independent of one another, so the total number of commands we have is 3^{12} .

b.) How many distinct commands can be created if word position does not affect meaning and a given word may appear at most once in a single command? (Thus, "abca bbac abbb" and "bbac abbb abca" should NOT count as different commands, and "aaca baaa aaca" is an INVALID command.)

Since we are not allowed to repeat words and word order doesn't matter, we are essentially choosing three words out of the a possible number of words. Thus, we must first figure out the possible number of words. There are three choices for each of four symbols, using the multiplication principle as we did in part a, we have $3^4 = 81$ possible Martian words. Of these, we can choose three to make a valid command. Thus, the total number of possible commands here is ${}_{81}C_3 = (81)(80)(79)/6 = 85320$

5) The greatest common divisor(gcd) of three positive integers a, b, and c is simply defined as the largest positive integer that is a factor of a, b, and c.

a) Show that the following is false: $\text{gcd}(a, b, c) = \text{gcd}(a, b+c)$.

Let $a=6, b=5, c=7$. Clearly $\text{gcd}(a,b,c) = 1$ since none of the values share a common factor. But, we have $\text{gcd}(a,b+c) = \text{gcd}(6,12) = 6$. Thus, the given statement is indeed false.

b) Describe a method (that does NOT use prime factorization of the three integers) to compute the gcd of three integers, and use this method to find $\text{gcd}(26, 78, 117)$.

$$\text{gcd}(a,b,c) = \text{gcd}(\text{gcd}(a,b),c).$$

Thus, we can use Euclid's algorithm to compute the gcd of a and b. Let this value be d. Then, reuse Euclid's algorithm to calculate the gcd of c and d. (Let this be e.) This value, is the desired gcd. This works because we know that $d|a, d|b, e|d$ and $e|c$. Since $e|d$, it follows that $e|a$ and $e|b$. This shows that our algorithm returns a common divisor of a, b and c. Now, we need to show that this is the greatest common divisor of the three values. Let f be a divisor of a, b and c. We will show that $f | e$, proving that e is the gcd. Surely, if $f|a$ and $f|b, f|d$, since d is the greatest common divisor of a and b. (All common divisors of a number divide into it's gcd.) But we also know that $f|c$. Thus, f is a common divisor of c and d. It follows that $f | e$, which is the $\text{gcd}(c,d)$.

Using the technique on the given values:

$$\text{gcd}(26, 78) = 26, \text{ since}$$

$$78 = 3 \times 26$$

$$\text{gcd}(26, 117) = 13$$

$$117 = 4 \times 26 + 13$$

$$26 = 2 \times 13$$

Thus, we have that $\text{gcd}(26, 78, 117) = 13$.

6)

a) Prove that a relation $R \subseteq A \times A$ is transitive if and only if $R^2 \subseteq R$.

First we will show that $R \subseteq A \times A$ is transitive $\Rightarrow R^2 \subseteq R$.

Let (x,y) be an arbitrary element of R^2 . We must show that it is also an element of R .

If $(x,y) \in R^2$, by the definition of relation composition, we have that there exists an element $z \in A$ such that $(x,z) \in R$ and $(z,y) \in R$. Since we are given that R is transitive, we can deduce that $(x,y) \in R$ as desired.

Now, let's show that $R^2 \subseteq R \Rightarrow R$ is transitive.

We must show that for any arbitrary elements x, y and z , if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.

If we have that $(x,y) \in R$ and $(y,z) \in R$, by definition of relation composition, it follows that $(x,z) \in R^2$. But, we are given that $R^2 \subseteq R$. So it necessarily follows that $(x,z) \in R$, which is what we needed to show.

b) Prove or disprove: If a relation $R \subseteq A \times A$ is symmetric then R^2 is symmetric as well.

We must show for arbitrary elements x and y , if $(x,y) \in R^2$, then $(y,x) \in R^2$.

If $(x,y) \in R^2$, by definition of relation composition, there exists an element $z \in A$ such that $(x,z) \in R$ and $(z,y) \in R$. Since R is symmetric, we know that $(y,z) \in R$, and $(z,x) \in R$. By using the definition of relation composition again, we have that $(y,x) \in R^2$, as desired.