Computer Science Foundation Exam

May 6, 2005

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: _________________________________

SSN: __________________________________

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone.

Credit cannot be given when your results are unreadable.

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work both of the following problems (1 and 2).**

1) Induction

Prove by induction: n^3-7n+3 is divisible by 3, $\forall n \in Z, n \geq 0$

2) Sets

Let A, B and C be arbitrary sets taken from the set of positive integers. Prove or disprove the following claims:

(a) if $B \cap C \subseteq A$, then $A - B \subseteq C$.

(b) If $A - (B \cap C) = A - B - C$, then $A \cap B = A \cap B \cap C$.

Solution to Problem 1:

Prove by induction: n^3-7n+3 is divisible by 3, $\forall n \in Z$, $n \geq 0$

Solution

Basis Step: **(5 pts)** When n=0, n^3-7n+3 = 0 + 0 + 3 = 3, which is clearly divisible by 3. Thus, the formula is true for n=0.

Inductive hypothesis: **(5 pts)** Assume for an arbitrary positive integer value of n=k that (k^3-7k+3) is divisible by 3.

Inductive step: Prove for n=k+1 that (($(k+1)^3$-7(k+1)+3) is divisible by 3.

$(k+1)^3-7(k+1)+3$ **(5 pts)**
$= (k^3+3k^2+3k+1) – 7k-4$
$= k^3+3k^2-4k-3$

rewrite as:

$=(k^3-7k+3) + (3k^2+3k-6)$
$=(k^3-7k+3) + 3*(k^2+k-2)$ **(5 pts)**

By the induction hypothesis, the first term is divisible by 3. The second term is divisible by 3. Hence the sum is divisible by 3. QED. **(5 pts)**

Solution to Problem 2:

Let A, B and C be arbitrary sets taken from the set of positive integers. Prove or disprove the following claims:

(a) if $B \cap C \subseteq A$, then $A - B \subseteq C$.

(b) If $A - (B \cap C) = A - B - C$, then $A \cap B = A \cap B \cap C$.

Solution

(a) The claim is false. Consider the following counter-example: $A = \{1\}$ $B = C = \emptyset$. In this situation we have $B \cap C = \emptyset$, so $B \cap C \subseteq A$ holds, but $A - B = \{1)$, thus, $A - B \subseteq C$ is false. **(10 pts)**

(b) This claim is true. We will prove it using proof by contradiction.

Assume to the contrary that $A \cap B \neq A \cap B \cap C$. Since it's clear that $A \cap B \cap C \subseteq A \cap B$, it must follow that there exists a positive integer x such that $x \in A \cap B$, but $x \notin A \cap B \cap C$. **(5 pts)** Since the latter logically reduces to $x \notin A$ or $x \notin B$ or $x \notin C$, and we know that $x \in A$ and $x \in B$, it follows that that $x \notin C$.

This infers that $x \notin B \cap C$, and since $x \in A$, it follows that $x \in A - (B \cap C)$. **(5 pts)**

Finally, notice that since $x \in B$, $x \notin A - B$, and clearly $x \notin A - B - C$, regardless of the contents of C. This contradicts the given information that $A - (B \cap C) = A - B - C$, completing the proof by contradiction. Thus, the only logical conclusion is that $A \cap B = A \cap B \cap C$ holds as desired. **(5 pts)**

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Section II B

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

Name: ________________________________

SSN: _________________________________

In this section of the exam, there are four (4) problems.

You must do <u>two</u> (2) of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone.

Credit cannot be given when your results are unreadable.

PART B: Work any two of the following problems (3 through 6).

3) Counting

(a) What is the number of ways to seat 10 guests in 10 chairs at a round table? Assume that arrangements are considered the same when one can be obtained from the other by rotation.

(b) Find the number of ways to seat these guests if two of the guests (guest1 and guest2) want to seat next to each other. Assume that arrangements are considered the same when one can be obtained from the other by rotation.

(c) The next day guest1 and guest2 want to seat next to each other and guest3 wants to seat opposite to guest2. Find the number of guest arrangements. Assume that arrangements are considered the same when one can be obtained from the other by rotation.

(d) The day after that only 5 out of 10 people show up. Count the number of ways to seat them in 10 chairs. Assume that arrangements are considered the same when one can be obtained from the other by rotation.

(e) One week later 10 other people show up, they are 5 married couples, and each couple would like to share two adjacent seats. Count the number of ways to seat them in 10 chairs at a round table. Assume that arrangements are considered the same when one can be obtained from the other by rotation.

<u>Solution</u>

(a) It's 9!. **(5 pts, 3pts if the answer is 10!)**

b) If guest1's position is fixed, there are two choices for guest2. After this we need to seat 8 people in 8 remaining seats, giving us 8! choices. Multiplying out, there are 1 x2 x8! ways. **(2 pts for the reason, 3 pts for the answer, use for c, d and e also!!!)**

c) If guest2's position is fixed, then 2 positions for guest1 and one for guest3. Finally, there are 7! ways to seat remaining 7 people. Thus the number of ways is 1x2x7!

d) If guest1's position is fixed, nine for the second, etc ... , six for the fifth person. Number of ways to seat five people is 1x 9x 8 x7x6:

e) Think of each married couple as a unit. If first unit's position is fixed, then there are 4! ways to match couples with pairs of seats. Since we must account for each married couple being able to switch seats with each other, this gives a factor of 2 for each couple. The total is $4!\text{x}2^5$:

4) Relations

Let A = {1, 2, 3, 4, ..., 100}. Define the function u(x) = the units digit of the positive integer x. For example u(37) = 7 and u(95) = 5.

(a) Let a relation R over A be defined as follows:
$R = \{ (a, b) \mid a \in A \wedge b \in A \wedge u(a^4) = u(b^4) \}$. Prove that R is an equivalence relation.

(b) How many equivalence classes does R have?

(c) How many elements are in each equivalence class of R?

Solution

(a) We must first show that R is reflexive, symmetric and transitive.

For all values $a \in A$, we find that $u(a^4) = u(a^4)$ since u is a well-defined function, thus the same input always produces a particular output value, thus $(a,a) \in R$. **(4 pts)**

Next, we should that R is symmetric. We must prove that if $(a,b) \in R$, then $(b,a) \in R$. Using the given information, it must be the case that $u(a^4) = u(b^4)$. Thus, it follows that $u(b^4) = u(a^4)$, which means $(b,a) \in R$, as desired. **(4 pts)**

Finally, we must show that R is transitive. Namely, we must prove if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$. Using the given information, we have $u(a^4) = u(b^4)$, and that $u(b^4) = u(c^4)$, since = is transitive, we have that $u(a^4) = u(c^4)$, from which we can conclude $(a,c) \in R$. **(4 pts)**

(b) Since the units digit of x^4 for any integer x is only dependent upon the units digit of x, we only have to calculate $0^4, 1^4, \ldots , 9^4$. These are 0, 1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, so that the different units digits we obtain are 0, 1, 5 and 6. Thus, R has 4 equivalence classes. **(6 pts)**

(c) Clearly, {1, 11, ..., 91} are all in the same equivalence class, etc. In particular, we note that 1^4, 3^4, 7^4, and 9^4, all end in 1, thus, there are 40 elements in this equivalence class. (This is because there are 10 numbers in A that end in 1, 3, 7 and 9 each.) Secondly, note that 2^4, 4^4, 6^4, and 8^4, all end in 6, so using the same logic, there are 40 elements in this equivalence class. Finally, there are 10 elements in the other two equivalence classes. Formally, we have the following:

$|[1]| = 40$, $|[2]| = 40$, $|[5]| = 10$, and $|[10]| = 10$. (Note: [x] stands for the equivalence class containing the element x.) **(7 pts)**

5) Logic

a) Use truth table to define whether or not $[p\to(q\to r)]\to[(p\to q)\to(p\to r)]$ is a tautology.

b) Let '?' be an unknown Boolean logical operator. The logical statement $((\neg p \vee \neg(p \wedge (x\wedge y\wedge z)))\to p) \vee ((r \wedge \neg r) \vee (q \vee F_0)$ is equivalent to [p'?'q]. Find the Boolean logical operator '?'

Solution

a) **(10 pts)**

p	q	r	$q\to r$	$p\to(q\to r)$	$p\to q$	$p\to r$	$(p\to q)\to(p\to r)$	$x\to y$
1	1	1	1	1	1	1	1	1
1	1	0	0	0	1	0	0	1
1	0	1	1	1	0	1	1	1
1	0	0	1	1	0	0	1	1
0	1	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1

b) Note: There are many solutions to this problem. Here is one solution

$((\neg p \vee \neg(p \wedge (x\wedge y\wedge z)))\to p) \vee ((r \wedge \neg r) \vee (q \vee F_0))$

$((\neg p \vee \neg(p \wedge (x\wedge y\wedge z)))\to p) \vee (F_0 \vee (q \vee F_0))$ Inverse **(1 pts)**

$((\neg p \vee \neg(p \wedge (x\wedge y\wedge z)))\to p) \vee (F_0 \vee q)$ Identity **(1 pts)**

$((\neg p \vee \neg(p \wedge (x\wedge y\wedge z)))\to p) \vee (q)$ Identity **(1 pts)**

$((\neg p \vee \neg p \vee \neg(x\wedge y\wedge z))\to p) \vee (q)$ DeMorgan's **(2 pts)**

$((\neg p \vee \neg(x\wedge y\wedge z))\to p) \vee (q)$ Idempotent **(2 pts)**

$(\neg(\neg p \vee \neg(x\wedge y\wedge z)) \vee p) \vee (q)$ Implication **(2 pts)**

$((p \wedge (x\wedge y\wedge z)) \vee p) \vee (q)$ DeMorgan's **(2 pts)**

$p\vee q$ Absorption **(3 pts)**

The Boolean logical operator is $\vee$ **(1 pts)**

6) Number Theory

(a) Prove that there exist an infinite number of prime numbers.

(b) Consider the first step in Euclid's Algorithm, for determining the greatest common divisor between two positive integers a and b, with a > b:

a = qb + r,

where q and r are integers with 0 ≤ r < b. Prove that r < a/2.

Solution

(a) Use proof by contradiction. Assume there are a finite number of primes. If so, we can list all prime numbers: p_1, p_2, ... , p_n. Now consider the positive integer $P = (\prod_{k=1}^{n} p_k) + 1$. **(5pts)** Either, it must be prime itself, or have a prime factor smaller than it. But, we find that none of the values p_1, p_2, ... , p_n could be a factor since each would leave a remainder of 1 when divided into P. This means that either the number itself is prime (and this number if larger than anyone on the original list), or another number, not on the list is a prime factor of P. Either way, we have contradicted the fact that we had a complete list of all the prime numbers **(8 pts),** thus there are an infinite number of primes.

(b) We are given that b > r. Furthermore, since a > b, it follows by the division algorithm that q > 0. **(3 pts)** Thus, we have the following:

a = bq + r
  ≥ b + r, since q > 0 and b is positive. **(3 pts)**
  > r + r, since b > r **(3 pts)**
  = 2r

Thus, if follows that a > 2r. We can divide this inequality by 2 to obtain the final result, namely that r < a/2. **(3 pts)**