Computer Science Foundation Exam

May 5, 2006

Section II B

KEY

DISCRETE STRUCTURES

PART B: Work any two of the following problems (3 through 6).

3) (PRF) Relations

Given the set $A = \{2, 3, 4, 8, 9, 12, 18\}$, define a relation T over A such that
$T = \{(a, b) | a \in A$ and $b \in A$ and ab is a perfect square, i.e., $ab = c^2$ for some integer $c\}$.

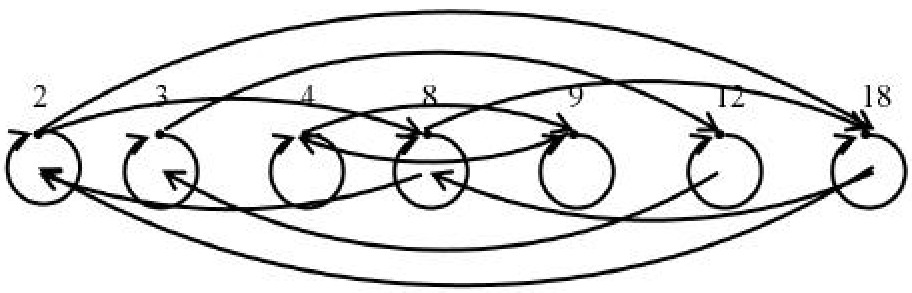
Answer the following questions.

a) (5 pts) Draw the directed graph representation of the relation T.

b) (15 pts) Determine, with proof, if the relation T satisfies each of the properties: reflexive, irreflexive, symmetric, anti-symmetric and transitive.

c) (5 pts) Give the matrix representation of T.

a)



Grading: 1 pt for all self loops, 1/2 point for the rest (approximately)

b) T is reflexive, because (2,2), (3,3), (4,4), (8,8), (9,9), (12,12) and (18,18) are all in T. (Note: In general, a relation defined like this would be reflexive because for all integers a, (a)(a) is a perfect square.) **(1 pt answer, 2 pts proof)**

T is not irreflexive since (2,2) is a member of T. **(1 pt answer, 2 pts proof)**

T is symmetric, clearly if (a)(b) is a perfect square, so is (b)(a), since multiplication is commutative. (We can also show this by listing out all the ordered pairs and showing that for all (a,b), (b,a) is also in the set.) **(1 pt answer, 2 pts proof)**

T is not anti-symmetric since (4,9) and (9,4) are members of T. **(1 pt answer, 2 pts proof)**

T is transitive. It's a bit exhausting to prove it. We only need to consider distinct elements a, b and c such that $(a,b) \in T$, and $(b,c) \in T$. Let's list these out: {(2,8), (2,18), (3, 12), (4,9), (8,18)} Notice that I don't bother listing out the symmetric pairs because, if this subset or elements of T is transitive, so will all of T. The only "matching" pair to check are (2,8) and (8,18). In fact, in this case, we see that (2,18) Is also indeed in T. Therefore T must be transitive.

(1 pt ans, 2 pts proof – allow some fairly weak proofs since this is only worth 2 pts.)

c)

$$\begin{array}{c c}
 & \begin{array}{ccccccc} 2 & 3 & 4 & 8 & 9 & 12 & 18 \end{array} \\
\begin{array}{c} 2 \\ 3 \\ 4 \\ 8 \\ 9 \\ 12 \\ 18 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}$$

Grading: about 1/4 pt per each 1 entry, round to the nearest integer out of 5.

4) (PRF) Functions

a) Let Z = {...−2, −1, 0, 1, 2, ...} denote the set of integers. Suppose f : Z→Z is a function, defined by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases}$$

i) (5 pts) Prove or disprove that f is one-to-one (injective)

ii) (10 pts) Prove or disprove that f is onto (surjective).

b) (10 pts) Let $f : A \to A$ be a function. Prove or disprove, that if f is onto, then $f \circ f$ is onto as well.

a-i) It can be disproved that f is one-to-one by the following counter-example. We have that $f(4)=f(1)=2$. **Grading: Any counter-example works, 2 pts for the answer, 3 points for the specific counter-example.**

a-ii) Proof. To prove that function f is onto, take arbitrary integer $k \in$ Z, to show that we can always find the pre-image of k under f, i.e. an integer n such that $f(n) = k$. **(4 pts)** Namely, take $n = 2k$, then, $f(2k) = k$, because $2k$ is even. **(6 pts)**

b) Proof. Assume that f is onto. To prove that $f \circ f$ is onto take arbitrary $y \in A$ to show that there exists some x such that $(f \circ f)(x) = y$. **(2 pts)** Since f is onto by assumption, we know, that there exists some $z \in A$, such that $f(z) = y$.**(2 pts)** Then by the onto property of f we can also imply that there exists $x \in A$ such that $f(x) = z$ **(2pts),** which means that we can find x such that $f(f(x)) = y$ **(2 pts)**. By the definition of the composite function it means that for arbitrary y we can find x such that $(f \circ f)(x) = y$, or the composite function $f \circ f$ is onto. (**2 pts)**

5) (CTG) Counting

a) (5 pts) How many permutations of the word ABRACADABRA are there?

The number of permutations is $\frac{11!}{5!2!2!1!1!}$. **Grading: numerator: 2 pts, Denominator: 3pts**

b) (10 pts) How many integers in the set {1, 2, 3, …, 1000} are divisible by 5 or 7?

Let set A be the set of values in the given set divisible by 5.
Let set B be the set of values in the given set divisible by 7.
$|A| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$, $|B| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$ **Grading: 2 pts for each of these.**

Note, since 5 and 7 are relatively prime, the intersection of A and B is simply the set of values divisible by 35. Thus, we have $|A \cap B| = \left\lfloor \frac{1000}{35} \right\rfloor = 28$. **Grading: 3 pts**

Using the inclusion-exclusion principle, we can now solve the problem:

$|A \cup B| = |A| + |B| - |A \cap B| = 200 + 142 - 28 = 314$ **Grading: 2pts formula, 1pt answer**

c) (10 pts) A class has 18 girls and 12 boys. In how many ways can a committee of two boys and two girls be chosen?

There are a total of $\binom{18}{2}$ ways to choose the girls. **Grading: 3 pts.**
There are a total of $\binom{12}{2}$ ways to choose the boys. **Grading: 3pts**.
Since the choice of each is independent of the other and the choices are combined to create a committee, we must use the multiplication principle to get the final answer of $\binom{18}{2}\binom{12}{2}$. **Grading: 4pts**

6) (NTH) Recursive Definitions and Number Theory

a) (20 pts) Use the Euclidean Algorithm to determine the greatest common divisor of 525 and 434. Then find integers x and y that satisfy the equation 525x + 434y = gcd(525,434).

525 = 1x434 + 91
434 = 4x91 + 70
91 = 1x70 + 21
70 = 3x21 + 7
21 = 3x7, so gcd(525, 434) = 7 **Grading: 1 pt per line, 2 pts for the final answer. (7 pts total)**

70 – 3x21 = 7
70 – 3x(91 – 70) = 7 **Grading: 3 pts to get to this substitution.**
4x70 – 3x91 = 7
4x(434 – 4x91) – 3x91 = 7 **Grading: Another 3 pts to get to here.**
4x434 – 19x91 = 7
4x434 – 19x(525 – 434) = 7 **Grading: 3 pts to here**
23x434 – 19x525 = 7 **Grading: 2 pts to here**
Thus, two integers x and y that satisfy the equation are x = -19, y = 23. **2 pts here**
(Grading: 13 pts total)

b) (5 pts) Determine the number of positive integer factors of $2^3 3^4 5^2$.

A positive factor of the term above must take the form $2^a 3^b 5^c$, where 0 = a = 3, 0 = b = 4, 0 = c = 2. Since the choices of a, b and c are independent of one another, and each unique value of a, b, and c corresponds to a unique factor of the given number, the total number of factors is equal to the total number of ordered triplets (a,b,c) that satisfy the restrictions given above. This number is simply (3+1)(4+1)(2+1) = 60, using the product rule.

Grading: No explanation necessary, give full credit if the answer is correct, no credit otherwise.