

# **Computer Science Foundation Exam Solutions**

**May 2, 2003**

**Section II A**

## **DISCRETE STRUCTURES**

**NO books, notes, or calculators may be used,  
and you must work entirely on your own.**

**Name:** \_\_\_\_\_

**SSN:** \_\_\_\_\_

**In this section of the exam, there are two (2) problems.**

**You must do both of them.**

**Each counts for 25% of the total exam grade.**

**Show the steps of your work carefully.**

**Problems will be graded based on the completeness of  
the solution steps and not graded based on the answer alone**

**Credit cannot be given when your results are**

**unreadable.**

## FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

### **PART A: Work both of the following problems (1 and 2).**

**1)** Given arbitrary sets A, B, C, and Y prove the following statement:

$$[A \times B \subseteq Y \subseteq C \times B] \rightarrow [((C - A) \times B) - \sim Y = Y - (A \times B)]$$

(Note:  $\sim$  denotes set complement, thus  $\sim Y$  is the complement of the set Y.)

**2)** Prove by induction that the sum of the squares of any six consecutive non-negative integers always leaves a remainder of seven when divided by twelve. (You may use induction in your solution, but are not required to do so.)

Solution to Problem 1:

Given arbitrary sets A, B, C, and Y prove the following statement:

$$[A \times B \subseteq Y \subseteq C \times B] \rightarrow [((C - A) \times B) - \sim Y = Y - (A \times B)]$$

(Note:  $\sim$  denotes set complement, thus  $\sim Y$  is the complement of the set Y.)

**Solution:**

**Assume if-clause is true:**

$$Y \subseteq C \times B \quad -- (1)$$
$$A \times B \subseteq Y \quad -- (2)$$

**Need to show two cases:**

- 1)  $((C - A) \times B) - \sim Y \subseteq Y - (A \times B)$
- 2)  $Y - (A \times B) \subseteq ((C - A) \times B) - \sim Y$

- 1) To show  $((C - A) \times B) - \sim Y \subseteq Y - (A \times B)$  we need to choose an arbitrary  $(x,y) \in ((C - A) \times B) - \sim Y$  and show that  $(x,y) \in Y - (A \times B)$ . By definition of relative complement and intersection we get:  $(x,y) \in (C - A) \times B$  and  $(x,y) \in Y$ . Also, since  $(x,y) \in (C - A) \times B$ , we get  $x \in (C - A)$  and  $y \in B$ . By definition of relative complement and intersection, we get  $x \in C$  and  $x \in \sim A$ . Since  $x \in \sim A$ , or  $x \notin A$ , it follows that  $(x,y) \notin (A \times B)$ , or  $(x,y) \in \sim(A \times B)$ . Therefore by definition of intersection and relative complement we get:  $(x,y) \in Y - (A \times B)$
- 2) To show  $Y - (A \times B) \subseteq ((C - A) \times B) \cap Y$  we need to choose an arbitrary  $(x,y) \in Y - (A \times B)$  and show that  $(x,y) \in ((C - A) \times B) - \sim Y$ . By definition of relative complement and intersection we get  $(x,y) \in Y$  and  $(x,y) \in \sim(A \times B)$ . Since  $Y \subseteq C \times B$  -- (1), it follows that  $x \in C$  and  $y \in B$ . Since  $(x,y) \in \sim(A \times B)$ , or  $(x,y) \notin (A \times B)$ . Since  $y \in B$ , it follows that  $x \notin A$ , or  $x \in \sim A$ . By Definition of intersection and relative complement, we get  $x \in (C - A)$ . Therefore  $(x,y) \in (C - A) \times B$  and by definition of intersection,  $(x,y) \in ((C - A) \times B) \cap Y$ . So by definition of relative complement we get  $(x,y) \in ((C - A) \times B) - \sim Y$

Grading: 5 points for stating what needs to be proven, 10 points for proving each case. (It probably makes more sense to do 3 and 11, but 5 and 10 seems to be a nicer split...)  
Within each case, 2 points for stating what needs to be proved, and 2 points for each following step.

Solution to Problem 2:

Prove by induction that the sum of the squares of any six consecutive non-negative integers always leaves a remainder of seven when divided by twelve. (You may use induction in your solution, but are not required to do so.)

**Solution:**

The sum of the squares of six consecutive non-negative integers can be represented as:

$$\sum_{i=n \dots n+5} i^2 \quad n \geq 0$$

**Basis:  $n = 0$**

$$\sum_{i=n \dots n+5} i^2 = \sum_{i=0 \dots 5} i^2 = 0^2 + 1^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

$$55 = 12(4) + 7 \quad \text{So remainder is 7 and the base case holds.}$$

**Induction Hypothesis:**

Assume that  $\sum_{i=k \dots k+5} i^2$  for some  $k \geq 0$  leaves a remainder of 7 when divided by 12.

**Inductive Step:**

Show that  $\sum_{i=k+1 \dots (k+1)+5} i^2$  leaves a remainder of 7 when divided by 12.

This sequence can be written out as:

$$\begin{aligned} & \dots k^2 + (k+1)^2 + (k+2)^2 + (k+3)^2 + (k+4)^2 + (k+5)^2 \dots \\ & \dots (k+1)^2 + (k+2)^2 + (k+3)^2 + (k+4)^2 + (k+5)^2 + (k+6)^2 \dots \end{aligned}$$

$$\text{So, } \sum_{i=k+1 \dots (k+6)} i^2 = \sum_{i=k \dots k+5} i^2 - k^2 + (k+6)^2$$

By the induction hypothesis, we know  $\sum_{i=k \dots k+5} i^2$  leaves remainder of 7 when divided by 12. Anything added to this would have to be divisible by 12 in order for the entire expression to leave a remainder of 7. Since,

$$-k^2 + (k+6)^2 = k^2 - k^2 + 12k + 36 = 12k + 36 = 12(k+6) \text{ and } 12 \mid 12(k+6)$$

leaves a remainder of 0 when divided by 12, then the entire expression  $\sum_{i=k \dots k+5} i^2 + [-k^2 + (k+6)^2]$  leaves a remainder of 7.

Therefore, the sum of the squares of any six non-negative integers ( $\sum_{i=n \dots n+5} i^2 \quad n \geq 0$ ) leaves a remainder of 7 when divided by 12.

**Alternate solution not using induction:**

**Given that  $n$  is an arbitrary integer, we can represent the sum of the squares of 6 consecutive integers as follows:**

$$\begin{aligned} & (n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 \\ &= n^2 - 4n + 4 + n^2 - 2n + 1 + n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 + n^2 + 6n + 9 \\ &= 6n^2 + 6n + 19 \\ &= 6n(n+1) + 12 + 7 \end{aligned}$$

**Since one of  $n$  and  $n+1$  must be even, the other must be odd. Thus, their product is always even. Let  $n(n+1) = 2c$ , where  $c$  is an integer and substitute:**

$$\begin{aligned} &= 6(2c) + 12 + 7 \\ &= 12c + 12 + 7 \\ &= 12(c+1) + 7 \\ &\equiv 7 \pmod{12} \text{ as desired.} \end{aligned}$$

Grading: For induction solution, 2 points for the stating the base case, the inductive hypothesis and the inductive step. 3 points from writing the initial sum, 5 points for recognizing this is the old sum minus the first element and plus the new element, the rest of the points are for the algebra