# Discrete Structures Foundation Exam Part A Solutions 3/9/01

**PART A**: **Work both of the following problems (1 and 2).**

**1)** Let $A$, $B$, and $C$ be sets such that $C \subset B$ (i.e., $C$ is a proper subset of $B$, or possibly $C = B$). Use appropriate set theoretic laws and theorems to prove that $(A - B) \cup (B - C) = \neg C \cap (A \cup B)$. Be sure to explain each step of your proof.

**(Solution one)**

LHS $= (A \cap \neg B) \cup (B \cap \neg C)$, by the definition of set difference(–)

$= (A \cup B) \cap (A \cup \neg C) \cap (\neg B \cup B) \cap (\neg B \cup \neg C)$, by the distributive law

$= (A \cup B) \cap (A \cup \neg C) \cap U \cap (\neg B \cup \neg C)$, by the inverse law

$= (A \cup B) \cap (A \cup \neg C) \cap (\neg B \cup \neg C)$, by the identity law

$= (A \cup B) \cap (A \cup \neg C) \cap \neg(B \cap C)$, DeMorgan's law

$= (A \cup B) \cap (A \cup \neg C) \cap \neg C$, the assumption $C \subset B$ implies $B \cap C = C$

$= (A \cup B) \cap \neg C$, by the absorption law $X \cap (X \cup Y) = X$

$= \neg C \cap (A \cup B)$, the commutative law

$=$ RHS

**(Solution two)**

RHS $= (\neg C \cap A) \cup (\neg C \cap B)$, by the distributive law

$= (A \cap \neg C) \cup (B \cap \neg C)$, the commutative law

$= (A - C) \cup (B - C)$, by the law $X - Y = X \cap \neg Y$

In order to prove that RHS = LHS, it suffices to prove

$(A - B) \cup (B - C) = (A - C) \cup (B - C)$

That is, we need to prove

$(A - B) \subset (A - C) \cup (B - C)$ --- (1) and

$(A - C) \subset (A - B) \cup (B - C)$ --- (2)

To prove (1), let $x \in A - B$. Thus, $x \in A$ and $x \notin B$.
Since $C \subset B$, so $x \notin C$. Thus, $x \in A - C$, which implies $x \in (A - C) \cup (B - C)$. So (1) is proved.
To prove (2), let $x \in A - C$. Thus, $x \in A$ and $x \notin C$ --- (3)
There are two cases: $x \in B$ or $x \notin B$.
In the first case, using (3) we have $x \in B - C \subset (A - B) \cup (B - C)$.
In the second case, using (3) we have $x \in A - B \subset (A - B) \cup (B - C)$.
Therefore, we proved (2) in both cases.

**2)** Prove by induction that for $n \geq 1$,

$$1(1!) + 2(2!) + \ldots + n(n!) = (n + 1)! - 1,$$

where the notation ! means factorial, i.e., $1! = 1$, and for $k \geq 2$, $k! = k\,(k - 1)!$.

**(Solution)**
(Basis) Consider $n$ = 1. In this case, the LHS = 1(1!) = 1; and
RHS = (1+1)! – 1 = 2! –1 = 2 – 1 = 1.
So LHS = RHS, which proves the Basis Step.
(Induction Hypothesis) Consider $n = k$. We assume

$$\sum_{j=1}^{k} j(j!) = (k+1)! - 1, \text{ for some } k \geq 1.$$

(Induction Step) Consider $n = k + 1$. We need to prove

$$\sum_{j=1}^{k+1} j(j!) = (k+1+1)! - 1 = (k+2)! - 1 \text{ --- (1)}$$

Note that LHS of $(1) = \sum_{j=1}^{k} j(j!) + (k+1)(k+1)!$, by the definition of summation

$$\begin{aligned}
&= (k+1)! - 1 + (k+1)(k+1)!, \text{ by the induction hypothesis} \\
&= (k+1)!\,(1 + (k+1)) - 1 \\
&= (k+1)!(k+2) - 1 \\
&= (k+2)! - 1, \text{ by the definition of !} \\
&= \text{RHS of } (1)
\end{aligned}$$

Thus, the induction step is proved. By induction we proved

$$\sum_{j=1}^{n} j(j!) = (n+1)! - 1, \text{ for all } n \geq 1.$$

# Discrete Structures Foundation Exam Part B Solutions 3/9/01

**PART B: Work any two of the following problems (3 through 6)**.

**3)** An ice cream shop lets its customers create their orders. Each customer can choose up to four scoops of ice cream from 10 different flavors. In addition, they can add any combination of the 7 toppings to their ice cream. (Note: Please leave your answer in factorials, combinations, and powers.)

a) If a customer is limited to at most two scoops of the same flavor, how many possible orders with exactly 4 scoops and up to 5 toppings can the customer make? (Assume each order has at least one topping.)

b) Suzanne wants to make 7 separate orders for ice cream. Each order will have exactly 1 scoop and 1 topping. If no flavor or topping is requested more than once, how many combinations of orders can Suzanne make?

**(Solution)**
a) If we ignore toppings initially, we have a problem of combinations WITH repetition. We are choosing 4 items from 10 possible items, allowing for repetition. This can be done in C(4+10-1,4) = 715. BUT, here we are counting choices that have 3 and 4 scoops of the same flavor. We need to subtract these out. So, our next sub-problem becomes to count the number of ways we can order exactly 4 scoops with one flavor repeated at least 3 times. Since only ONE flavor can be repeated at least 3 times, pick this flavor. There are 10 choices for it. Go ahead and pick 3 scoops of this flavor. Now you are left with 1 scoop to pick out of the 10 total flavors. This can be done in 10 ways as well. Thus, there are a total of 10x10 = 100 combinations of scoops with one flavor repeated at least 3 times. So we have 715 – 100 = 615 ways to choose the scoops of ice cream.

Now, the choice of toppings is independent from the scoops. There are total of $2^7$ total combinations of toppings we can receive without restrictions. BUT, we are only allowed to get up to 5 toppings, but at least one topping. Thus we just subtract out the number of ways to get 0, 6 or 7 toppings. There is C(7,0) = 1 way to get zero toppings, C(7,6) = 7 ways to choose 6 toppings, and C(7,7)=1 way to choose all 7 toppings. So there are a total of $2^7 - 1 - 7 - 1 = 119$ ways to choose the toppings.
This gives us a final answer of 615x119 = 73185 possible orders for the customer.

b) This question is the same as how many injections are there from a set of size 7 to a set of size 10. Imagine the domain being the toppings. Since we are forced to pick each topping exactly once, and none of the flavors are repeated, we are mapping each topping to a distinct element from the co-domain, the set of flavors. We can do this is P(10,7) ways. P(10,7) = 10!/3! = 604800.

**4)** Let $R$ be a relation with $R \subset A \times A$, where $|A| = 5$. Answer the following questions, giving justifications for your answers.
a) Give an example of a non-empty relation that is both symmetric and anti-symmetric. (Let the set $A = \{a, b, c, d, e\}$.)
b) How many relations $R$ are symmetric?

**(Solution)**
a) R = {(a,a)}, any non-empty subset of {(a,a), (b,b), (c,c), (d,d), (e,e)} will suffice.

b) For a relation to be symmetric, when a≠b, if (a,b)∈R, then we MUST have (b,a)∈R. So, in creating a symmetric relation R, of the 20 ordered pairs of the form (a,b) with a≠b, we can REALLY only choose whether C(5,2) = 10 of them are in the relation or not. This can be done in $2^{10}$ ways. Furthermore, we have a choice as to whether any of the ordered pairs of the form (a,a) are in the relation. There are 5 such ordered pairs. So, we have $2^5$ choices here. Multiplying these, we find that there are a total of $2^{10} \times 2^5 = 2^{15} = 32768$ total symmetric relations.

**5)** Let $f$: $A \rightarrow B$, be a function, and $g$: $B \rightarrow C$ be a function.

a) If $g \circ f : A \rightarrow C$ is surjective (onto), prove that g is also surjective (onto).

b) Give a small example of functions $f$ and $g$ such that $f$ is an injection (one-to-one) and $g$ is a surjection (onto), but $g \circ f$ is not an injection (one-to-one).

**(Solution)**

a) Our goal is to prove that g is surjective. We must show the following:

For all $c \in C$, there exists a $b \in B$ such that g(b) = c.

It is good enough to show that for an arbitrarily chosen element c from the set C, there exists an element b from the set B such that g(b) = c.

We are given that g°f is surjective. Thus, we know that there exists some element $a \in A$ such that g(f(a)) = c. BUT, f IS a function. Thus, we know that f(a) = b, for some element from the set B. So we now have that g(b) = c. But this is all we wanted to prove. We have found an element in the set b such that g(b) = c.

b) Let the set A = {1,2}, B={a,b,c}, C={x,y}

f(1) = a
f(2) = c
g(a) = x
g(b) = y
g(c) = x
g(f(1)) = g(f(2)) = x

**6)** Let $D(n)$ be the $n$ digit number consisting of just 1s. Use induction (or any other method) to show that $11 \mid D(2n)$, for all integers $n > 0$. (Note that we can recursively define $D(n)$ as follows: $D(n) = 10D(n - 1) + 1$ for $n > 1$, $D(1) = 1$. Explicitly, $D(n) = (10^n - 1) / 9$.)

**(Solution #1 – Direct)**:

$$\begin{aligned} D(2n) = (10^{2n} - 1)/9 &= (10^2 - 1)(10^{2(n-1)} + 10^{2(n-2)} + \ldots 1)/9 \\ &= 99(10^{2(n-1)} + 10^{2(n-2)} + \ldots 1)/9 \\ &= 11(10^{2(n-1)} + 10^{2(n-2)} + \ldots 1) \end{aligned}$$

Since the value in the parentheses is an integer, it follows that D(2n) is divisible by 11.

**(Solution #2 – Induction)**:

Use induction on n.
Base Case: n=1. D(2x1) = 11. 11 | 11 thus, the base case holds.

Now, assume that for an arbitrary n=k, that 11|D(2k).
We must prove, under this assumption, for n=k+1, that 11|D(2(k+1)).

$$\begin{aligned}
D(2(k+1)) &= D(2k+2) \\
&= 10D(2k+1) + 1 \\
&= 10(10D(2k) + 1) + 1 \\
&= 100D(2k) + 10 + 1 \\
&= 100D(2k) + 11
\end{aligned}$$

Using our inductive hypothesis, we have that 11 | D(2k). By definition, this means that there exists an integer c such that D(2k) = 11c. Now we have

$$\begin{aligned}
D(2(k+1)) &= 100D(2k) + 11 \\
&= 100(11c) + 11 \\
&= 11(100c + 1)
\end{aligned}$$

From here, since we know that 100c+1 is an integer, by the definition of divisibility, we can conclude that 11| D(2(k+1)) as desired.