Part I. Choose exactly one question (Question 1 or Question 2) in this part to answer.

1. Let A, B, and C denote 3 sets. Prove that if $A \cup C = B \cup C$ and $A \cap C = B \cap C$, then $A = B$. (**Hint:** Let $x \in A$, then $x \in A \cup C$ is true.)
   We first prove $A \subset B$. Let $x \in A$ ---- (1), we need to prove $x \in B$ ---- (2).
   From (1), we have $x \in A \cup C = B \cup C$. Thus, $x \in B$ or $x \in C$. There are two cases:
   (Case one) Suppose $x \in B$. In this case, (2) is already true.
   (Case two) Suppose $x \in C$. In this case, (1) implies that $x \in A \cap C = B \cap C \subset B$. Thus, $x \in B$ is also true.
   Therefore, we proved (2) in both cases.
   The proof for $B \subset A$ is similar, due to the symmetry of A and B. Therefore, we proved $A = B$.
2. Let $n \geq 1$ denote a positive integer. Answer the following two parts for this question:
   (a) Prove without using induction that

$$\frac{1}{2^n+1} + \frac{1}{2^n+2} + \ldots + \frac{1}{2^n+2^n} \geq \frac{1}{2}.$$

   (**Hint** : Explain $\frac{1}{2^n+i} \geq \frac{1}{2^n+2^n}$ for each i in the range $1 \leq i \leq 2^n$.)

   Note that for each i in the range $1 \leq i \leq 2^n$, we have $2^n + i \leq 2^n + 2^n$. Therefore,

$$\frac{1}{2^n+i} \quad \frac{1}{2^n+2^n} \text{ and thus, } \frac{1}{2^n+1} + \frac{1}{2^n+2} + \ldots + \frac{1}{2^n+2^n} \quad 2^n\left(\frac{1}{2^n+2^n}\right) = \frac{1}{2}.$$

   (b) Use the results of Part (a) and use induction on $n \geq 1$ to prove the following inequality:

$$\sum_{j=1}^{2^n} \frac{1}{j} \geq 1 + \frac{n}{2}.$$

   (Basis Step) When $n = 1$. In this case,

$$\text{LHS} = {}_{j=1}^{2^n} \frac{1}{j} = {}_{j=1}^{2} \frac{1}{j} = 1 + \frac{1}{2}, \text{ and RHS} = 1 + \frac{n}{2} = 1 + \frac{1}{2} = \text{LHS}.$$

   Thus, the Basis Step is proved.
   (Induction Hypothesis) When $n = k$. Suppose

$${}_{j=1}^{2^k} \frac{1}{j} \quad 1 + \frac{k}{2} \text{ for some } k \quad 1.$$

   (Induction Step) When $n = k + 1$. We need to prove

$${}_{j=1}^{2^{k+1}} \frac{1}{j} \quad 1 + \frac{k+1}{2} \text{ ---- (1).}$$

$$\text{Note that LHS of (1)} = \sum_{j=1}^{2^k} \frac{1}{j} + \sum_{j=2^k+1}^{2^{k+1}} \frac{1}{j}, \text{ by t finition of summation}$$

$$1 + \frac{k}{2} + \sum_{j=2^k+1}^{2^{k+1}} \frac{1}{j}, \text{ by the Induction Hypothesis}$$

$$1 + \frac{k}{2} + \frac{1}{2}, \text{ using the results of Part (a)}$$

$$= 1 + \frac{k+1}{2}$$

Therefore, the Induction Step is proved. By induction, we have proved the inequality for all $n \geq 1$.

Part II. Choose exactly one question (Question 3 or Question 4) in this part to answer.

3. Let $f : A \to B$ and $g : B \to C$ denote two functions. Answer the following two parts for this question:
   (a) If both f and g are injections, then prove the composition function $g \circ f : A \to C$ is an injection.
   Let $g \circ f(x) = g \circ f(y)$ ---- (1), we need to prove $x = y$ ---- (2).
   From (1), $g(f(x)) = g(f(y))$ ---- (3), by the definition of function composition. Since g is an injection by assumption, (3) implies $f(x) = f(y)$, which then implies $x = y$ because f is an injection by assumption. Thus, (2) is proved.
   (b) If the function $g \circ f : A \to C$ is a surjection, and g is an injection, then prove the function f is a surjection. (This part is independent of Part (a).)
   To prove $f: A \to B$ is a surjection, let $y \in B$ ---- (1), then we need to find $x \in A$ such that $f(x) = y$ ---- (2). From (1), $g(y) \in C$. Because the function $g \circ f : A \to C$ is a surjection by assumption, there exists $x \in A$ such that $g \circ f(x) = g(y)$. Therefore, $g(f(x)) = g(y)$ ---- (3). Since g is an injection by assumption, so (3) implies $f(x) = y$, which proves (2).
4. Let W, T, and Y denote 3 sets of strings over an alphabet A. Answer the following two parts for this question:
   (a) Prove that $(WT)^* \cup (WY)^* \subset (W(T \cup Y))^*$.
   Since $T \subset T \cup Y$, so $WT \subset W(T \cup Y)$, which implies $(WT)^* \subset (W(T \cup Y))^*$ ---- (1).
   Similarly, we can prove $(WY)^* \subset (W(T \cup Y))^*$ ---- (2).
   Combining (1) and (2) yields $(WT)^* \cup (WY)^* \subset (W(T \cup Y))^*$.
   (b) Use a "small" example to show that $(W(T \cup Y))^* \subset (WT)^* \cup (WY)^*$ is **false**.
   Let $A = \{a, b\}$, $W = \{\lambda\}$, $T = \{a\}$, $Y = \{b\}$. Then,
   $(W(T \cup Y))^* = \{a, b\}^*$ = the set of all strings over symbols a and b.
   In particular, the string $ab \in (W(T \cup Y))^*$.
   However, since $(WT)^* = \{a\}^*$ and $(WY)^* = \{b\}^*$, we have
   $(WT)^* \cup (WY)^* = \{a\}^* \cup \{b\}^*$,
   thus, $ab \notin (WT)^* \cup (WY)^*$. Therefore, $(W(T \cup Y))^* \subset (WT)^* \cup (WY)^*$ is false.