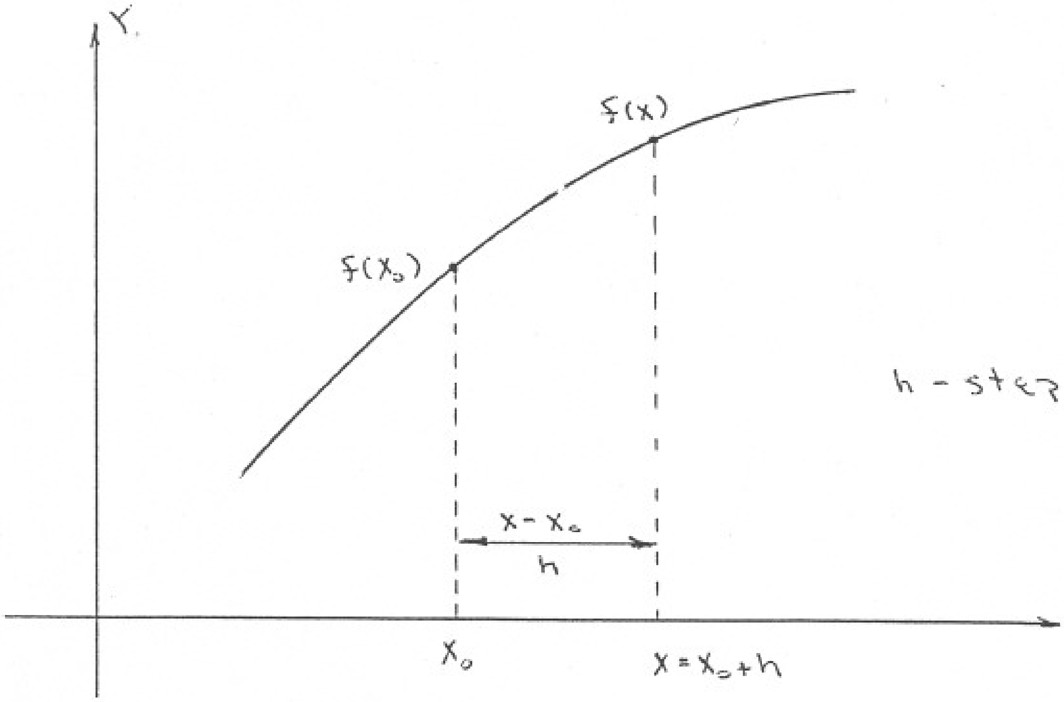
The Taylor Series

The Taylor Series Expansion of a function $f(x)$ about the point x_0 is given by

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \text{-------}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$



h - step size

Let $h = x - x_0 \Rightarrow x = x_0 + h$

$$\Rightarrow f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)h^2}{2!} + \frac{f'''(x_0)h^3}{3!} + \text{-------}$$

Alternate form of Taylor Series Expansion

The n^{th} order approximation of $f(x)$ at $x = x_0 + h$ is .

$$f_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots\cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$(n = 0, 1, 2, 3, \cdots\cdots)$

For $n = 0, 1$ and 2

$f_0(x) = f(x_0)$ Zeroth Order Taylor Series approximation

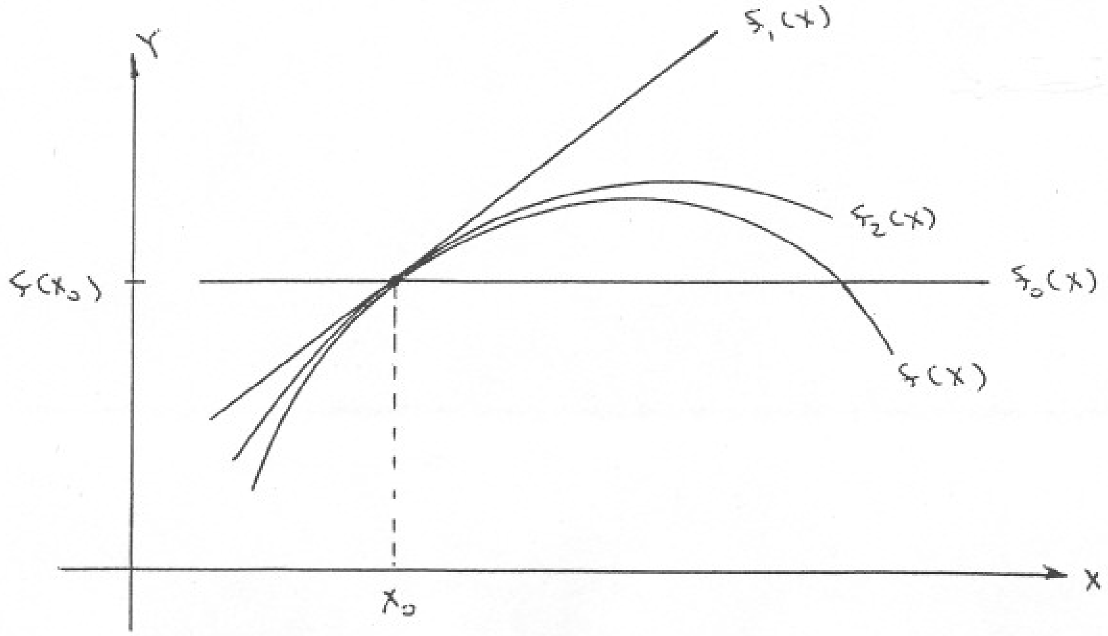
$f_1(x) = f(x_0) + f'(x_0)(x-x_0)$ First order TS approximation

$\quad = f_0(x) + f'(x_0)(x-x_0)$

$f_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$ Second Order TS approximation

$\quad = f_1(x) + \frac{f''(x_0)}{2!}(x-x_0)^2$

Considering x as an independent variable, graphs of $f(x)$ and the truncated TS approximations of order 0, 1 and 2 are shown below.



Consider the 1^{st} order truncated TS approximation of $f(x)$ about the point x_0,

$$Y = f_1(x)$$
$$= f(x_0) + f'(x_0)(x - x_0) \qquad \text{Linear Function}$$

At $x = x_0$, $Y = f(x_0) + f'(x_0)(x_0 - x_0)$
$$= f(x_0)$$

Thus the point $\{x_0, f(x_0)\}$ is on the linear function $Y = f_1(x)$.

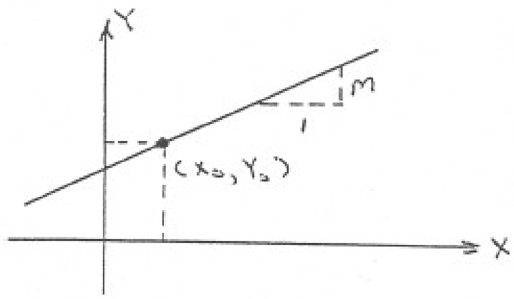
The slope of the line $Y = f_1(x)$ is constant and equal to

$$\frac{dY}{dx} = f'(x_0)$$

Alternatively, $Y = f_1(x)$ in point-slope form

$$Y - Y_0 = m(x - x_0)$$
$$Y - f(x_0) = f'(x_0)(x - x_0)$$

$\Rightarrow \quad m = f'(x_0)$



Since $f'(x_0)$ is the slope of the tangent to $Y = f(x)$ at $x = x_0$, the line $Y = f_1(x)$, i.e. the 1^{st} order TS approximation of $f(x)$ about the point x_0, must be the tangent line drawn to $Y = f(x)$ at $x = x_0$.

Suppose x is fixed or chosen to be a constant value, e.g. x_1. The <u>truncation</u> <u>error</u> resulting from using the zeroth, first and second order approximations of $f(x_1)$ are:

$n = 0, \quad E_0 = f(x_1) - f_0(x_1)$

$$E_0 = f(x_1) - f(x_0)$$

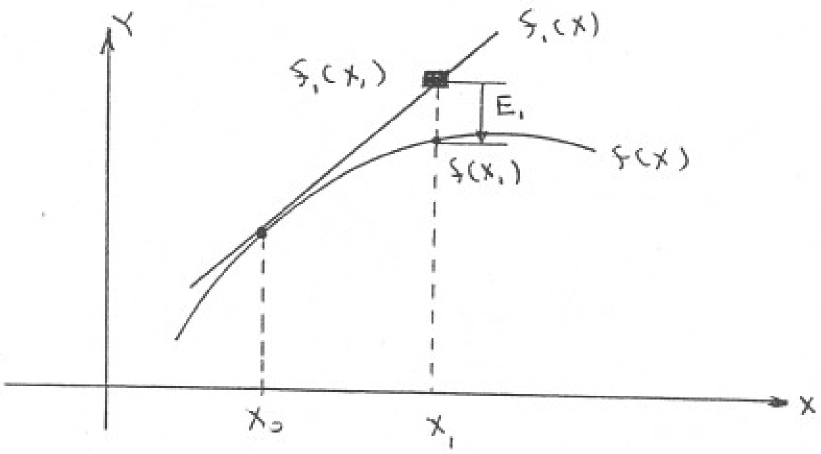
$n=1, \quad E_1 = f(x_1) - f_1(x_1)$

$$= f(x_1) - \left[f(x_0) + f'(x_0)(x_1 - x_0) \right]$$

$n=2, \quad E_2 = f(x_1) - f_2(x_1)$

$$= f(x_1) - \left[f(x_0) + f'(x_0)(x_1 - x_0) + \frac{f''(x_0)}{2!}(x_1 - x_0)^2 \right]$$

The error E_1 is illustrated below.



The Taylor Series expansion of $f(x)$ about x_0 can be broken down into components, i.e.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \text{-----} + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$+ \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1} + \frac{f^{(n+2)}(x_0)}{(n+2)!}(x - x_0)^{n+2} + \text{-------}$$

$$= f_n(x) + R_n$$

where R_n is the <u>remainder</u> term and is given by

$$R_n = \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1} + \frac{f^{(n+2)}}{(n+2)!}(x_0)(x - x_0)^2 + \text{------}$$

For $n = 1$,

$$f(x) = f_1(x) + R_1$$

Suppose x is chosen to be some constant x_1,

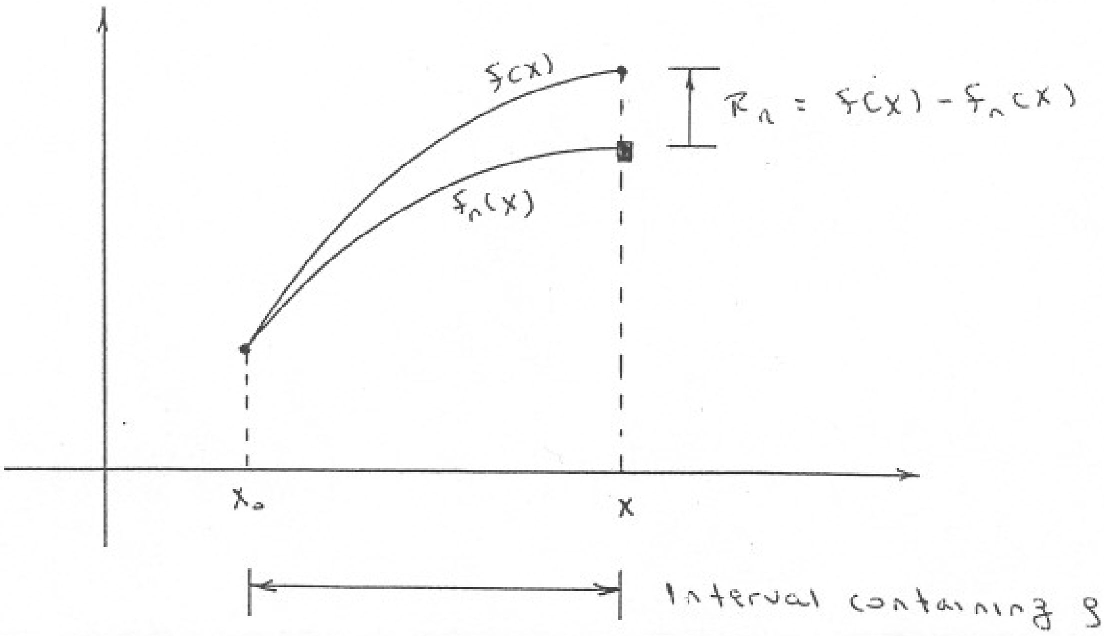
$$\Rightarrow \quad f(x_1) = f_1(x_1) + R_1$$

$$R_1 = f(x_1) - f_1(x_1)$$

In other words, the remainder R_1 is identical to the truncation error E_1 previously defined.

From Calculus, the Remainder Theorem states

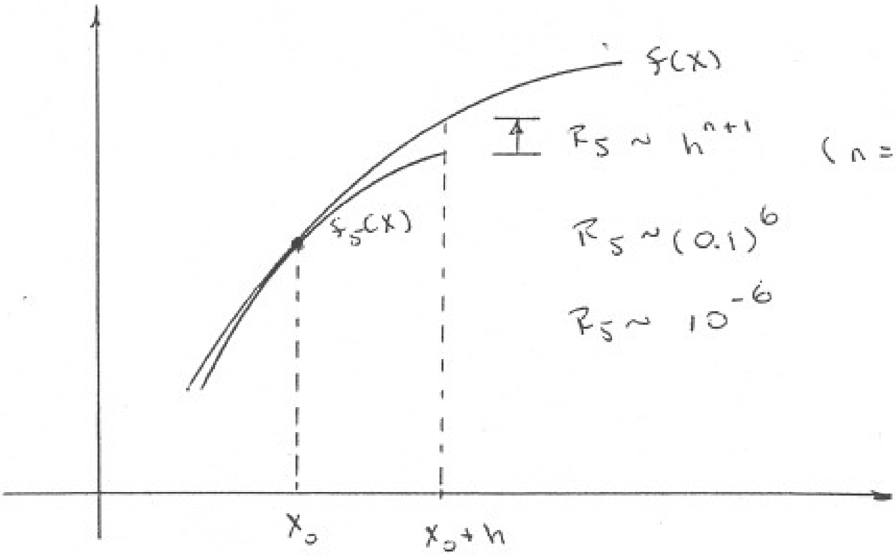
$$R_n = \frac{f^{(n+1)}}{(n+1)!}(\xi)(x - x_0)^{n+1} \qquad x_0 < \xi < x$$



It is very difficult to find the value of ξ when the function $f(x)$ is given and the points x_0 & x as well as the order of the truncated TS series n are all known.

The real value of the Remainder Theorem is the way it expresses the truncation error ($E_n = R_n$) in terms of the step size $h = x - x_0$ and the order of the truncated series n.

For example, suppose we have been using a 5th order truncated TS expansion about x_0 to approximate $f(x)$ at the point $x = x_0 + h$, where $h = 0.1$



$R_5 \sim h^{n+1}$ $(n = 5, h = 0.1)$

$R_5 \sim (0.1)^6$

$R_5 \sim 10^{-6}$

if we try different order truncated series with $h = 0.1$,

$\Rightarrow$ $R_4 \sim h^{4+1}$ $\qquad$ $R_6 \sim h^{6+1}$

$R_4 \sim (0.1)^5$ $\qquad$ $R_6 \sim (0.1)^7$

$R_4 \sim 10^{-5}$ $\qquad$ $R_6 \sim 10^{-7}$

$R_4 \sim 10 \times 10^{-6}$ $\qquad$ $R_6 \sim \frac{1}{10} \times 10^{-6}$

If the step size h changes with $n = 5$

$R_5 \sim \left(\frac{h}{2}\right)^{5+1}$ $\qquad$ $R_5 \sim (2h)^{5+1}$

$R_5 \sim \frac{1}{2^6} \times 10^{-6}$, $h = 0.05$ $\qquad$ $R_5 \sim 2^6 \times 10^{-6}$, $h = 0.2$

The Taylor Series Expansion of the function

$f(x) = e^x$ $(-\infty < x < \infty)$ about the point $x_0 = 0$ is

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + ----$$

$f(x_0) = e^{x_0}$, $f(0) = 1$

$f'(x_0) = e^{x_0}$, $f'(0) = 1$

$f''(x_0) = e^{x_0}$, $f''(0) = 1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\Rightarrow \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ----$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$ (T.S. Expansion of e^x about $x_0 = 0$)

For $x = 0$,

$$\Rightarrow \quad e^0 = 1 + 0 + 0 + ----$$

For $x = 0.5$

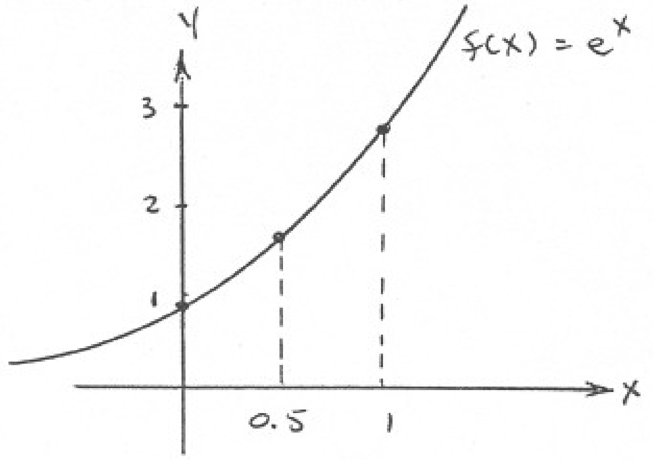
$$\Rightarrow \quad e^{0.5} = 1 + 0.5 + \frac{0.5^2}{2!} + \frac{0.5^3}{3!} + ----$$

The sum of the infinite series above is 1.648721271

For $x = 1$

$$\Rightarrow \quad e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + -----$$

The sum of the infinite series above is 2.718281828



$f(0) = e^0 = 1$

$f(0.5) = e^{0.5} = 1.648721271$

$f(1) = e^1 = 2.718281828$

Suppose we use the truncated Taylor series Expansion of e^x about $x_0 = 0$ to estimate $f(1) = e^1 = 2.718281828$

The first several truncated series are

$n=0, \quad f_0(x) = 1$

$n=1, \quad f_1(x) = 1+x$

$n=2, \quad f_2(x) = 1 + x + \frac{x^2}{2!}$

$n=3, \quad f_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$n=4, \quad f_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

Evaluating the above truncated series at $x=1$ in order to approximate $f(1) = e^1 = 2.718281828$

$\Rightarrow \quad f_0(1) = 1$

$f_1(1) = 1+1 = 2$

$f_2(1) = 1 + 1 + \frac{1^2}{2!} = 2.5$

$f_3(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} = 2.666666667$

$f_4(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708333333$

The <u>true error</u> resulting from using the 4^{th} order truncated TS expansion to approximate $f(1) = e^1$ is

$$E_T = f(1) - f_4(1)$$

$$= e^1 - 2.708333333$$

$$= 0.009948495$$

The true error is identical to the truncation error which is the sum of the remaining terms when $f_4(1)$ is used to estimate $f(1)$.

In general, the true error resulting from using an n^{th} order truncated Taylor Series expansion $f_n(x)$ to approximate the true $f(x)$ is

$$E_T = f(x) - f_n(x)$$

ex. Find the true error E_T when using $f_3(x)$ about $x_0 = 0.5$ to approximate $f(x) = e^x$ at $x = 1$.

$$f'(x_0) = e^{x_0} = e^{0.5}$$

$$f''(x_0) = e^{x_0} = e^{0.5}$$

$$f'''(x_0) = e^{x_0} = e^{0.5}$$

$$f_3(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$= e^{0.5} + e^{0.5}(x-0.5) + \frac{e^{0.5}}{2}(x-0.5)^2 + \frac{e^{0.5}}{6}(x-0.5)^3$$

$$\Rightarrow f_3(1) = e^{0.5} + e^{0.5}(1-0.5) + \frac{e^{0.5}}{2}(1-0.5)^2 + \frac{e^{0.5}}{6}(1-0.5)^3$$

$$= 2.713520425$$

$$E_T = f(1) - f_3(1) = 2.718281828 - 2.713520425 = 0.004761403$$

ex. Estimate $f(x) = \frac{1}{x}$ at $x = 1.25$ by using the truncated Taylor Series about $x_0 = 1$ and fill in the table.

n	$f(1.25)$	$f_n(1.25)$	E_T
0			
1			
2			
3			

$f(x) = \frac{1}{x}$, $f(x_0) = f(1) = 1$

$f'(x) = \frac{-1}{x^2}$, $f'(x_0) = f'(1) = -1$

$f''(x) = \frac{2}{x^3}$, $f''(x_0) = f''(1) = 2$

$f'''(x) = \frac{-6}{x^4}$, $f'''(x_0) = f'''(1) = -6$

$f_0(x) = f(x_0) = 1$

$f_1(x) = f(x_0) + f'(x_0)(x - x_0) = 1 - 1(x-1)$

$$f_2(x) = f_1(x) + \frac{f''(x_0)(x-x_0)^2}{2!} = 1 - 1(x-1) + \frac{2(x-1)^2}{2!}$$

$$= 1 - (x-1) + (x-1)^2$$

$$f_3(x) = f_2(x) + \frac{f'''(x_0)(x-x_0)^3}{3!} = 1-(x-1)+(x-1)^2 - \frac{6}{3!}(x-1)^3$$

$$= 1-(x-1)+(x-1)^2-(x-1)^3$$

$f(1.25) = \left.\frac{1}{x}\right|_{x=1.25} = 0.8$ $\quad (n = 0, 1, 2, 3)$

$n = 0,$ $\quad f_0(1.25) = 1$

$$E_T = f(1.25) - f_0(1.25)$$
$$= 0.8 - 1$$
$$= -0.2$$

$n = 1,$ $\quad f_1(1.25) = 1 - (1.25 - 1) = 0.75$

$$E_T = f(1.25) - f_1(1.25)$$
$$= 0.8 - 0.75$$
$$= 0.05$$

$n = 2,$ $\quad f_2(1.25) = 1 - (1.25 - 1) + (1.25 - 1)^2$

$$= 0.75 + 0.0625$$
$$= 0.8125$$

$$E_T = f(1.25) - f_2(1.25)$$
$$= 0.8 - 0.8125$$
$$= -0.0125$$

$n = 3,$ $\quad f_3(1.25) = 1 - (1.25 - 1) + (1.25 - 1)^2 - (1.25 - 1)^3$

$$= 0.8125 - 0.015625$$
$$= 0.796875$$

$$E_T = f(1.25) - f_3(1.25)$$
$$= 0.8 - 0.796875$$
$$= 0.003125$$

The true relative error is given by

$$e_T = \frac{E_T}{\text{true value}} = \frac{\text{true value} - \text{approximate value}}{\text{true value}}$$

In the previous example, the true relative errors are

$$e_T = \frac{f(1.25) - f_n(1.25)}{f(1.25)}$$

$n=0$, $$e_T = \frac{0.8 - 1}{0.8} = -0.25 \quad (-25\%)$$

$n=1$, $$e_T = \frac{0.8 - 0.75}{0.8} = 0.0625 \quad (6.25\%)$$

$n=2$, $$e_T = \frac{0.8 - 0.8125}{0.8} = -0.015625 \quad (-1.5625\%)$$

$n=3$, $$e_T = \frac{0.8 - 0.796875}{0.8} = 0.00390625 \quad (0.390625\%)$$

Both the true error, E_T, and the true relative error, e_T, can only be evaluated when the true value is known.

The approximate relative error refers to the fractional change of an estimate between two consecutive iterations. That is,

$$e_A = \frac{\text{new value} - \text{old value}}{\text{new value}}$$

In the example using an n^{th} order truncated Taylor Series to approximate a function value at some "x", the approximate relative error from using $f_n(x)$ to estimate $f(x)$ is

$$e_A = \frac{f_n(x) - f_{n-1}(x)}{f_n(x)} , \quad n = 1, 2, 3, -----$$

In the previous example,

$n = 1$, $$e_A = \frac{f_1(1.25) - f_0(1.25)}{f_1(1.25)}$$

$$= \frac{0.75 - 1}{0.75} = -0.333333 \quad (-33.3333\%)$$

$n = 2$, $$e_A = \frac{f_2(1.25) - f_1(1.25)}{f_2(1.25)}$$

$$= \frac{0.8125 - 0.75}{0.8125} = 0.0769231 \quad (7.69231\%)$$

$n = 3$, $$e_A = \frac{f_3(1.25) - f_2(1.25)}{f_3(1.25)}$$

$$= \frac{0.796875 - 0.8125}{0.796875} = -0.0196078 \quad (-1.96078\%)$$

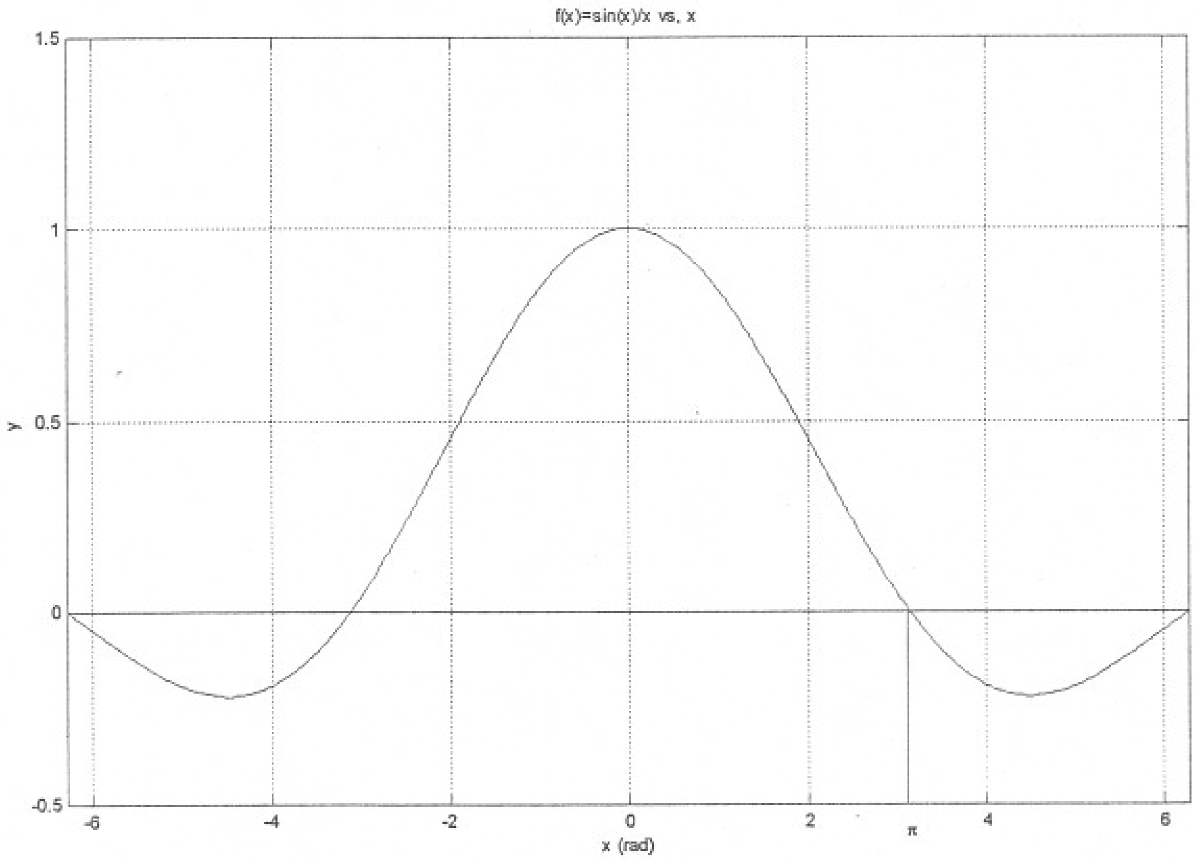
EGN 3420 Exam I Name______________

Fa 2000

SHOW ALL WORK!

Problem 1 (30pts)

For the function $f(x) = \dfrac{\sin x}{x}$ shown below.



a) Find $f_2(x)$, the second order truncated Taylor Series expansion of $f(x)$ about $x_0 = \pi$.

b) Use $f_2(x)$ to approximate $f(0)$.

c) Defining $f(0)$ as

$$\lim_{x \to 0} f(x) = \lim_{x \to 0}\left[\frac{\sin x}{x}\right]$$

Use L'Hopital's rule to find $f(0)$ and the true error E_T at x=0.

$$f(x) = \frac{\sin x}{x}$$

a) $$f_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$f''(x) = \frac{x^2\left[x(-\sin x) + \cos x - \cos x\right] - (x\cos x - \sin x)(2x)}{x^4}$$

$$= \frac{-x^3 \sin x - 2x^2 \cos x + 2x \sin x}{x^4}$$

$$x_0 = \pi$$

$$f(x_0) = \frac{\sin \pi}{\pi} = 0$$

$$f''(x_0) = \frac{\pi \cos \pi - \sin \pi}{\pi^2} = \frac{-1}{\pi}$$

$$f''(x_0) = \frac{-\pi^3 \sin \pi - 2\pi^2 \cos \pi + 2\pi \sin \pi}{\pi^4} = \frac{2}{\pi^2}$$

$$f_2(x) = 0 - \frac{1}{\pi}(x-\pi) + \frac{1}{2}\frac{2}{\pi^2}(x-\pi)^2$$

$$= -\frac{1}{\pi}(x-\pi) + \frac{1}{\pi^2}(x-\pi)^2$$

b) $$f_2(0) = \frac{-1}{\pi}(0-\pi) + \frac{1}{\pi^2}(0-\pi)^2$$

$$= 1 + 1$$

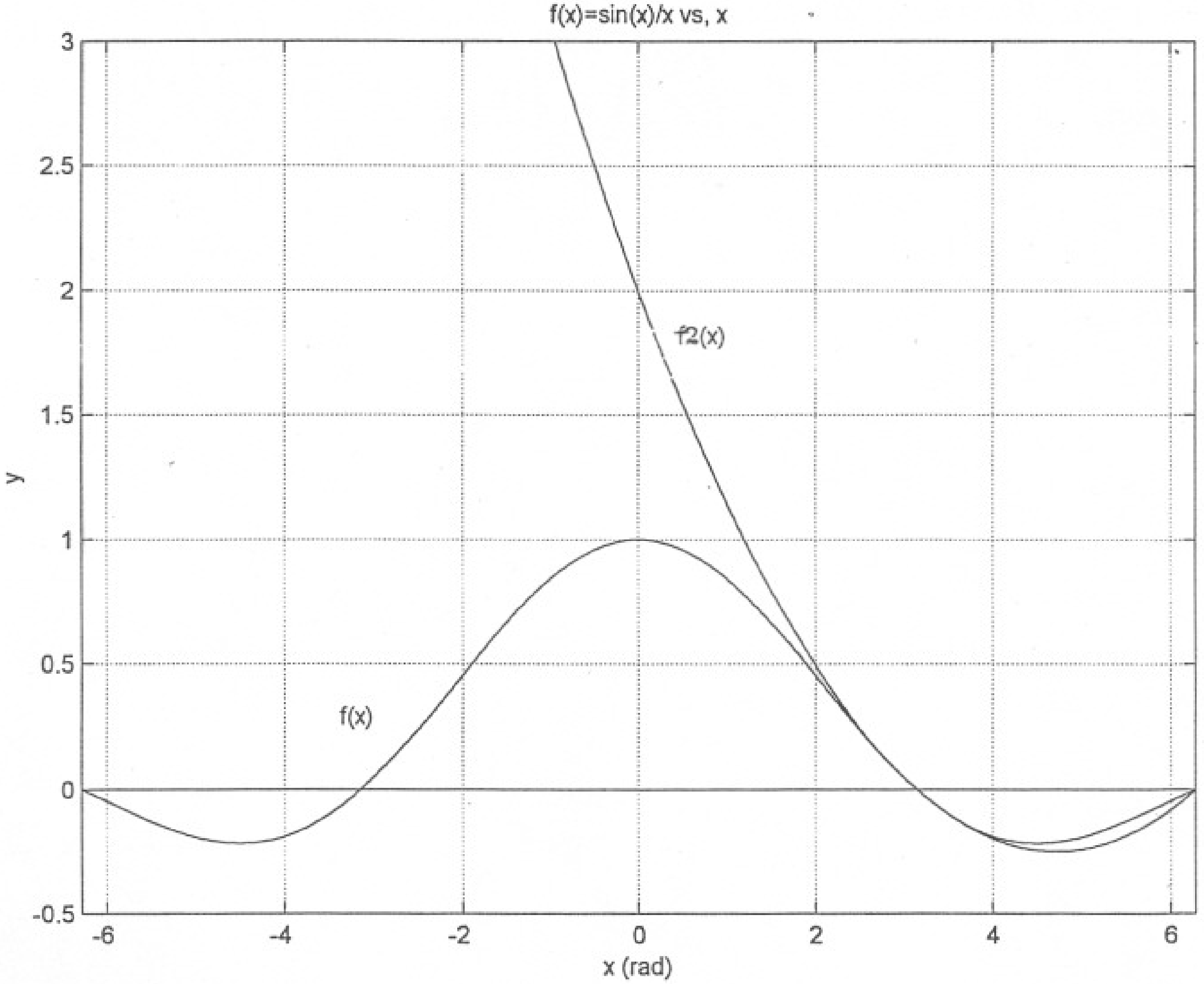
$$= 2$$

c) $f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$

$E_T = f(0) - f_2(0)$

$= 1 - 2$

$= -1$



Write a MATLAB program to do the following:

a) Find the truncated Taylor Series expansions $f_n(x)$, $n = 0,1,2,...,9,10$ about the point $x_0 = 0$ for $f(x) = \cos x$. Graph $f_n(x)$, $n = 0,1,2,4,6,8,10$ along with $f(x)$ over the interval $0 \le x \le 2\pi$.

b) Use $f_n(x)$, $n = 0,1,2,...,9,10$ to approximate $f(\pi/3) = \cos\pi/3 = 0.5$ by filling in the table below. Express all answers to 14 places (format "long" in MATLAB) after the decimal point.

n	$f(\pi/3) = \cos(\pi/3)$	$f_n(\pi/3)$	$E_T = f(\pi/3) - f_n(\pi/3)$
0	0.50000000000000	1.00000000000000	-0.50000000000000
1	0.50000000000000	1.00000000000000	-0.50000000000000
2	0.50000000000000	0.45168864438392	0.0483113 5561608
3	0.50000000000000	0.45168864438392	0.0483113 5561608
4	0.50000000000000	0.50179620150018	-0.00179620150018
5	0.50000000000000		
6	0.50000000000000		
7	0.50000000000000		
8	0.50000000000000		
9	0.50000000000000		
10	0.50000000000000		

c) Repeat Steps 1 and 2 for $x_0 = \pi / 4$.

n	$f(\pi / 3)$	$f_n(\pi / 3)$	$E_T = f(\pi / 3) - f_n(\pi / 3)$
0	0.50000000000000	0.70710678118655	-0.20710678118655
1	0.50000000000000	0.52198665876328	-0.02198665876328
2	0.50000000000000	0.49775449140343	0.00224550859657
3	0.50000000000000	0.49986914693004	0.00013085306995
4	0.50000000000000		
5	0.50000000000000		
6	0.50000000000000		
7	0.50000000000000		
8	0.50000000000000		
9	0.50000000000000		
10	0.50000000000000		

d) Repeat Steps 1 and 2 for $x_0 = \pi / 2$.

n	$f(\pi / 3)$	$f_n(\pi / 3)$	$E_T = f(\pi / 3) - f_n(\pi / 3)$
0	0.50000000000000	0.00000000000000	0.50000000000000
1	0.50000000000000	0.52359877559830	-0.02359877559830
2	0.50000000000000	0.52359877559830	-0.02359877559830
3	0.50000000000000	0.49967417939436	0.00032582060563
4	0.50000000000000		
5	0.50000000000000		
6	0.50000000000000		
7	0.50000000000000		
8	0.50000000000000		
9	0.50000000000000		
10	0.50000000000000		

Project 1 - Taylor Series Expansion of $f(x) = \cos x$

$f(x) = \cos x$, $\quad f(x_0) = \cos x_0$

$f'(x) = -\sin x$, $\quad f'(x_0) = -\sin x_0$

$f''(x) = -\cos x$, $\quad f''(x_0) = -\cos x_0$

$f'''(x) = \sin x$, $\quad f'''(x_0) = \sin x_0$

$f^{iv}(x) = \cos x$, $\quad f^{iv}(x_0) = \cos x_0$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$f_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \cdots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

For $x_0 = 0$,

$$f_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$\Rightarrow \quad f_0(x) = 1$

$$f_1(x) = 1 + (-0)\,x = 1$$

$$f_2(x) = 1 + \frac{(-1)}{2!}x^2 = 1 - \frac{x^2}{2!}$$

$$f_3(x) = 1 - \frac{x^2}{2!} + \frac{(0)}{3!}x^3 = 1 - \frac{x^2}{2}$$

$$f_4(x) = 1 - \frac{x^2}{2!} + \frac{(1)}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

Therefore, the n^{th} order truncated Taylor Series of $f(x) = \cos x$ about $x_0 = 0$ is given by

$$f_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{----} + \frac{(-1)^{n/2}}{n!} x^n \quad (n=0,2,4,\text{----})$$

For $x = \pi/3$, $f(x) = f(\pi/3) = \cos \pi/3 = 0.5$

$f_0(\pi/3) = 1$, $E_T = f(\pi/3) - f_0(\pi/3)$

$= 0.5 - 1$

$= -0.5$

$$f_2(\pi/3) = 1 - \frac{(\pi/3)^2}{2!} = 0.45168864$$

$E_T = f(\pi/3) - f_2(\pi/3)$

$= 0.5 - 0.45168864$

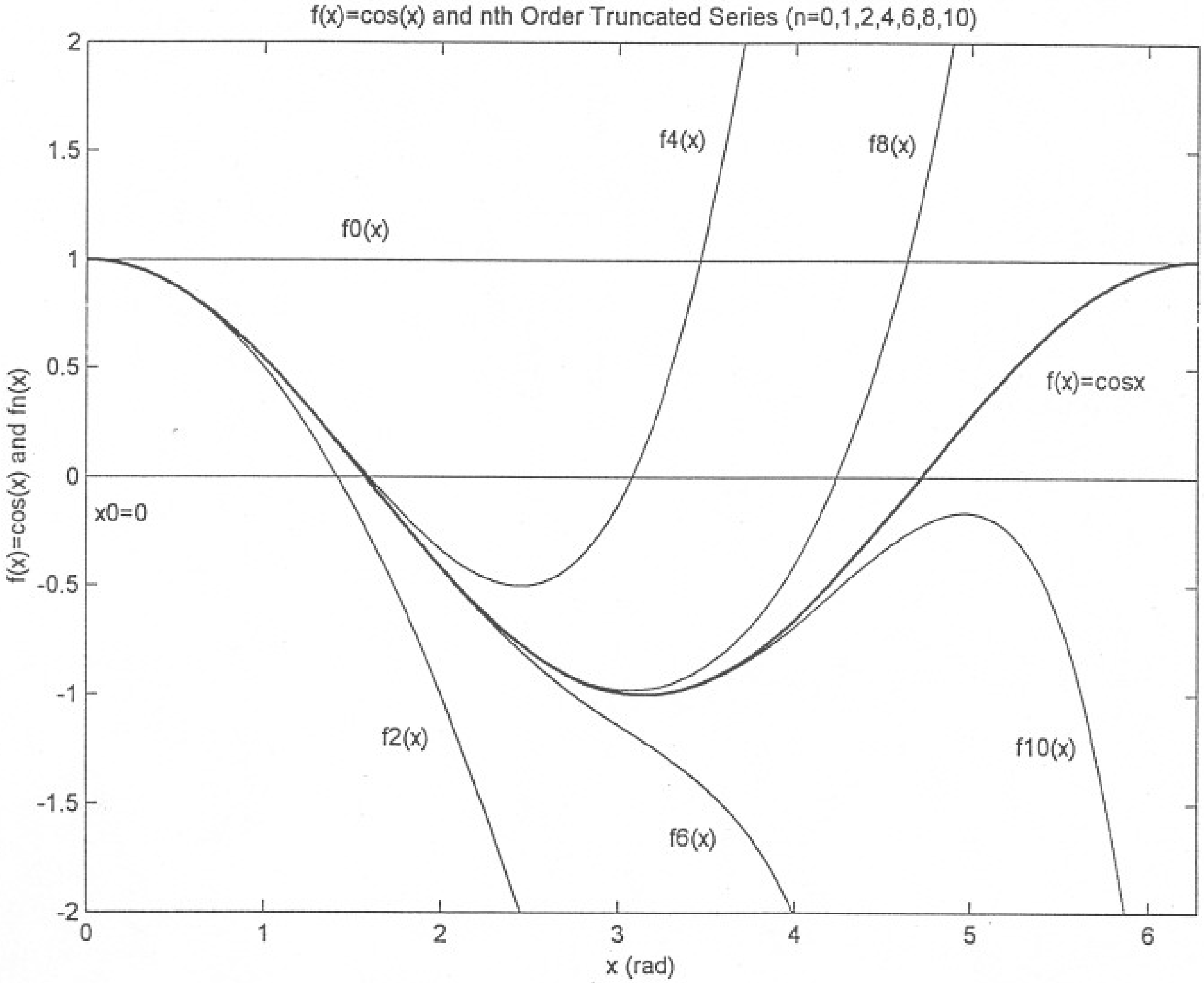
$= 0.04831136$

$$f_4(\pi/3) = 1 - \frac{(\pi/3)^2}{2!} + \frac{(\pi/3)^4}{4!} = 0.50179620$$

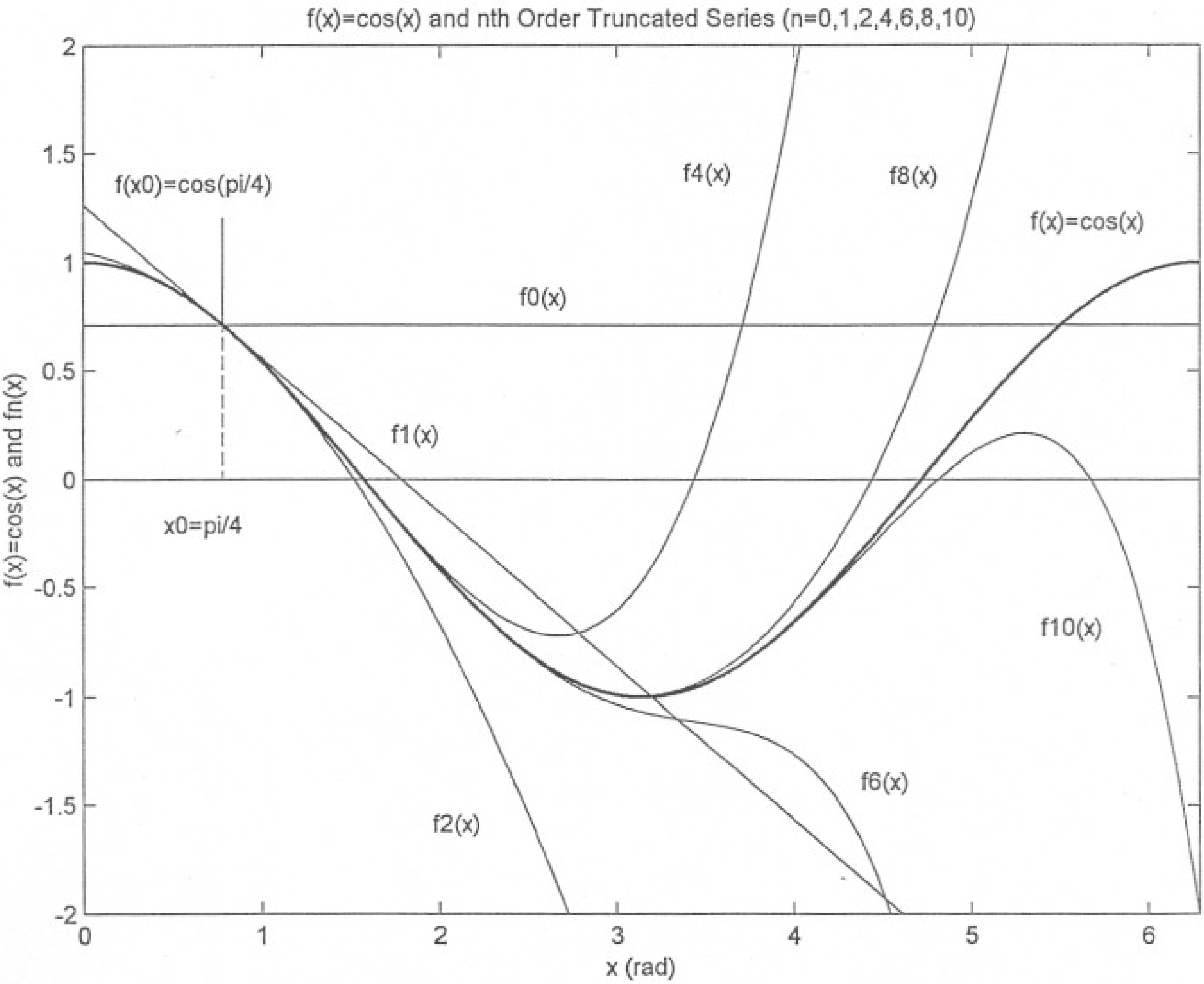
$E_T = f(\pi/3) - f_4(\pi/3)$

$= 0.5 - 0.50179620$

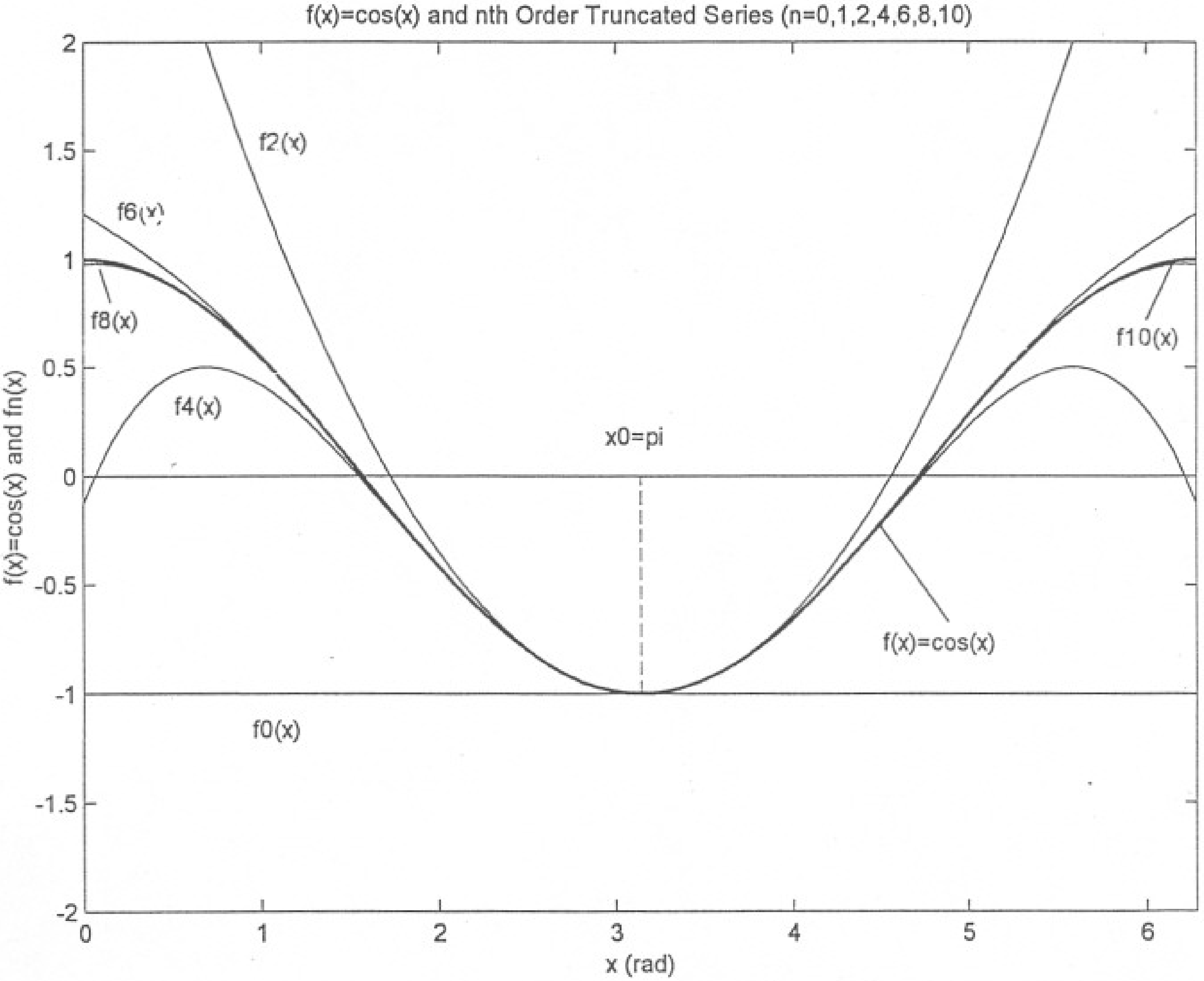
$= -0.00179620$



f(x)=cos(x) and nth Order Truncated Series (n=0,1,2,4,6,8,10)
f4(x)
f8(x)
f0(x)
f(x)=cosx
x0=0
f2(x)
f10(x)
f6(x)
f(x)=cos(x) and fn(x)
x (rad)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6



f(x)=cos(x) and nth Order Truncated Series (n=0,1,2,4,6,8,10)
f(x0)=cos(pi/4)
f4(x)
f8(x)
f(x)=cos(x)
f0(x)
f1(x)
x0=pi/4
f10(x)
f6(x)
f2(x)
f(x)=cos(x) and fn(x)
x (rad)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6


f(x)=cos(x) and nth Order Truncated Series (n=0,1,2,4,6,8,10)
f2(x)
f6(x)
f8(x)
f4(x)
f10(x)
x0=pi
f(x)=cos(x)
f0(x)
f(x)=cos(x) and fn(x)
x (rad)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6