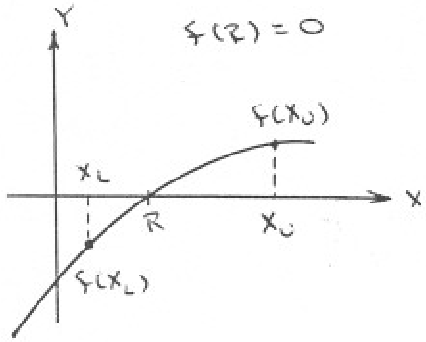
Root Solving

Bisection Method

The Bisection method begins with an interval (x_L, x_U) over which the function changes sign.

That is, $f(x_L)\,f(x_U) < 0$



The midpoint of the interval (x_L, x_U) is calculated and becomes the first iterated value, i.e. estimate of root R.

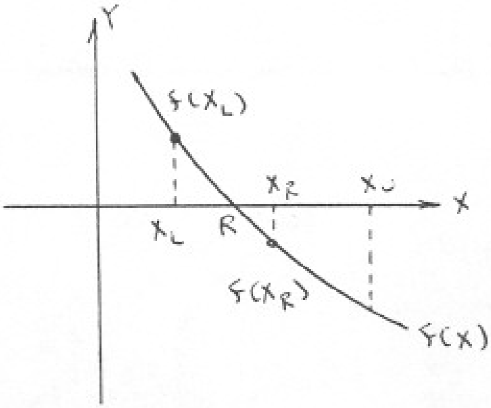
$$x_R = \frac{x_L + x_U}{2}$$

The next step is to calculate $f(x_R)$.

The signs of $f(x_L)$ and $f(x_R)$ determine whether the root R is located in the equal intervals (x_L, x_R) or (x_R, x_U).

If $f(x_L)$ and $f(x_R)$ have the same sign, then the root R lies in the interval (x_R, x_U).

If $f(x_L)$ and $f(x_R)$ have opposite signs, then the root R lies in the interval (x_L, x_R).

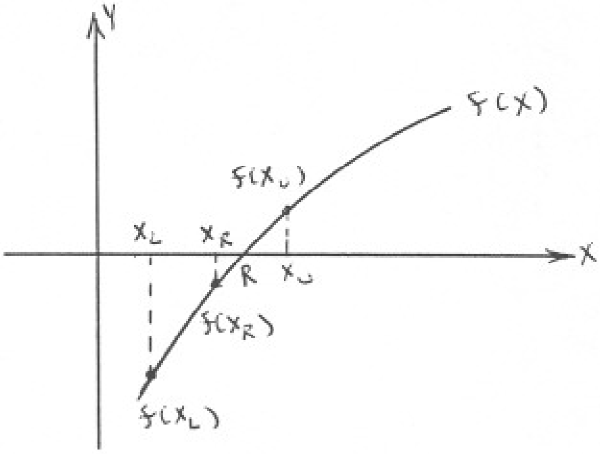


$f(x_L)\,f(x_R) = (+)(-) < 0$

$\Rightarrow R$ is in (x_L, x_R)

$x_U \leftarrow x_R$

Next Iteration Begins with New interval (x_L, x_U)



$f(X_L)\,f(X_R) = (-)(-) > 0$

$\Rightarrow$ R is in (X_R, X_U)

$X_L \leftarrow X_R$

Next Iteration Begins with New Interval (X_L, X_U)

The approximate relative error can be used as the basis for deciding when to stop the iterations.

$$|e_A| = \left| \frac{X_R^{new} - X_R^{old}}{X_R^{new}} \right| \times 100$$

$|e_A|$ is the per cent relative change (without regard to sign) in successive estimates of the root R.

The iterations stop when $|e_A|$ is less than some appropriately chosen "stop value" e_s.

choosing e_s according to

$$e_s = (0.5 \times 10^{2-n})\ \%$$

guarantees that the result, i.e. last iteration value, is correct to at least n significant figures.

n	e_s
1	5 %
2	0.5 %
3	0.05 %

ex. $f(x) = x^3 - 7x^2 + 10.99x - 4.95$

A root lies between $x_L = 0.75$ and $x_U = 1.0$

Use the Bisection Method to approximate the root and stop the iterations when $|e_A| < 1\%$.

Iteration	x_L	x_U	x_R	$f(x_R)$	$f(x_L)$	$\lvert e_A \rvert, \%$
1	0.75	1	0.875	-0.023	-0.223	—
2	0.875	1	0.9375	0.0247	-0.023	6.666
3	0.875	0.9375	0.90625	0.00495	-0.023	3.448
4	0.875	0.90625	0.890625	-0.00806	-0.023	1.754
5	0.890625	0.90625	0.89843	-0.00129	-0.00806	0.869

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Exam I

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SHOW ALL WORK!

Problem 2 (25pts)

The function $f(x)=\dfrac{\sin x}{x}$ has a root located between 2 and 4. Fill in the table below for the first three iterations of the Bisection Method. Express all answers to four digits after the decimal point.

Iteration	x_l	x_u	x_r	$f(x_l)$	$f(x_r)$	e_A, %	e_T, %
1	2.0000	4.0000	3.0000	0.4546	0.0470	—	4.5070
2	3.0000	4.0000	3.5000	0.0470	-0.1002	14.2857	-11.4085
3	3.0000	3.5000	3.2500	0.0470	-0.0333	-7.6923	-3.4507

$$x_r = \frac{x_l + x_u}{2} = \frac{2+4}{2} = 3$$

$$f(x_l) = \frac{\sin x_l}{x_l} = \frac{\sin 2}{2} = 0.4546$$

$$f(x_r) = \frac{\sin x_r}{x_r} = \frac{\sin 3}{3} = 0.0470$$

True Root $R = \pi$

$$e_T = \frac{R - x_r}{R} \times 100 = \frac{\pi - 3}{\pi} \times 100 =$$

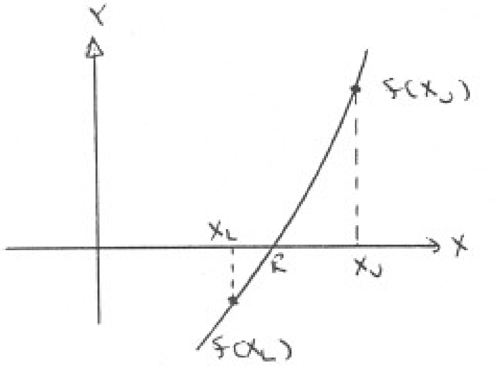
$$f(x_l)\, f(x_r) = (+)(+) > 0$$

$\Rightarrow$ $x_l = 3$, $x_u = 4$

$$e_A = \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \times 100 = \frac{3.5 - 3}{3.5} \times 100 = -11.4085$$

The False Position Method

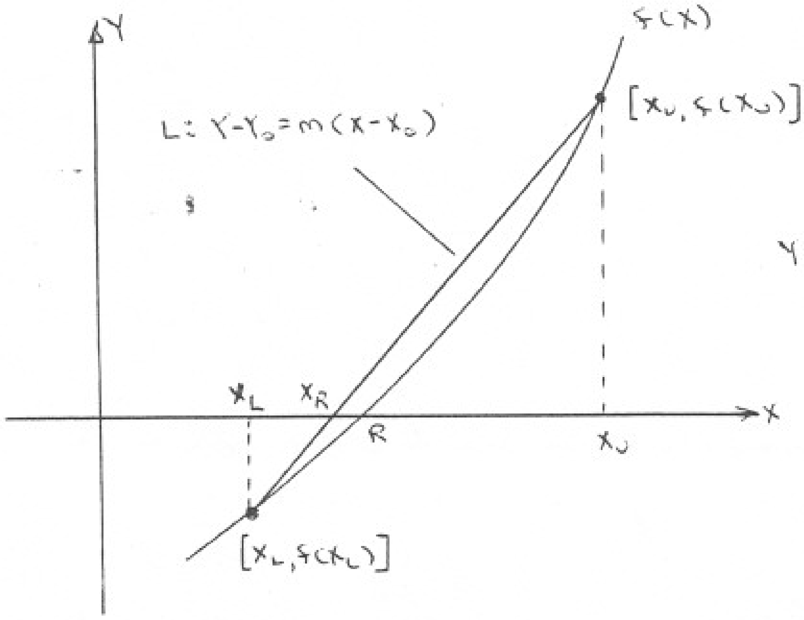
The Bisection Method does not take into account the magnitudes of $f(X_L)$ and $f(X_U)$. If $|f(X_L)|$ and $|f(X_U)|$ are an order of magnitude (or more) different, the root R is probably closer to the end at which the function value is smaller in magnitude.



$|f(X_U)| >> |f(X_L)|$

$\Rightarrow R$ is closer to X_L

The False Position Method estimates the root R to be at the intersection of the straight line segment connecting the points $[X_L, f(X_L)]$ and $[X_U, f(X_U)]$ with the x-axis.



$$m = \frac{f(X_U) - f(X_L)}{X_U - X_L}$$

$$Y - f(X_L) = \left[\frac{f(X_U) - f(X_L)}{X_U - X_L}\right](X - X_L)$$

Setting $Y = 0$ and solving for X (and calling it X_R)

$$\Rightarrow X_R = X_L - f(X_L)\left[\frac{X_U - X_L}{f(X_U) - f(X_L)}\right] \text{ OR } X_R = X_U - f(X_U)\left[\frac{X_U - X_L}{f(X_U) - f(X_L)}\right]$$

Open Methods

Open methods are based on formulas or algorithms that attempt to approach the true root without the use of brackets. Typically a single starting value is required and the method will either converge to the true root or diverge away from it as the iterations continue.

Simple Fixed Point Iteration

Given the problem of finding the root R of $f(X) = 0$ it is sometimes possible to rearrange the terms into the form $x = g(x)$.

ex. $f(X) = X^2 + 2X - 1 = 0$

$$X = \frac{1-2X}{X} \Rightarrow g(X) = \frac{1-2X}{X}$$

Also,

$$X = \frac{1-X^2}{2} \Rightarrow g(X) = \frac{1-X^2}{2}$$

If it is not possible to solve explicitly for X, then X can be added to both sides of $f(X) = 0$.

ex. $f(X) = e^{-X} \sin X = 0$

$$X + e^{-X} \sin X = X \Rightarrow g(X) = X + e^{-X} \sin X$$

An initial guess or estimate of the root is used to locate the next iterated value as follows:

$X_1 = g(X_0)$ where X_0 is the initial guess

X_1 is the result of the first iteration

Subsequent iterations are

$$x_2 = g(x_1)$$

$$x_3 = g(x_2)$$

$$\vdots \qquad \vdots$$

$$x_{i+1} = g(x_i)$$

ex. $f(x) = x^2 + 2x - 1$, $\quad R = -1 \pm \sqrt{2} = -2.41421356\,2$
$0.41421356\,2$

$$x = g(x) = \frac{1 - x^2}{2}$$

i	x_i	$E_T = R - x_i$	$\lvert e_T \rvert$, %	$\lvert e_A \rvert$, %
0	0	0.4142136	100	—
1	0.5	-0.08578864	20.71	100
2	0.375	0.0392136	9.47	33.33
3	0.4296875	-0.0154739	3.74	12.73
4	0.4076843	0.0065293	1.58	5.42
5	0.4168968	-0.0026832	0.65	2.21
6	0.4130986	0.001115	0.27	0.92

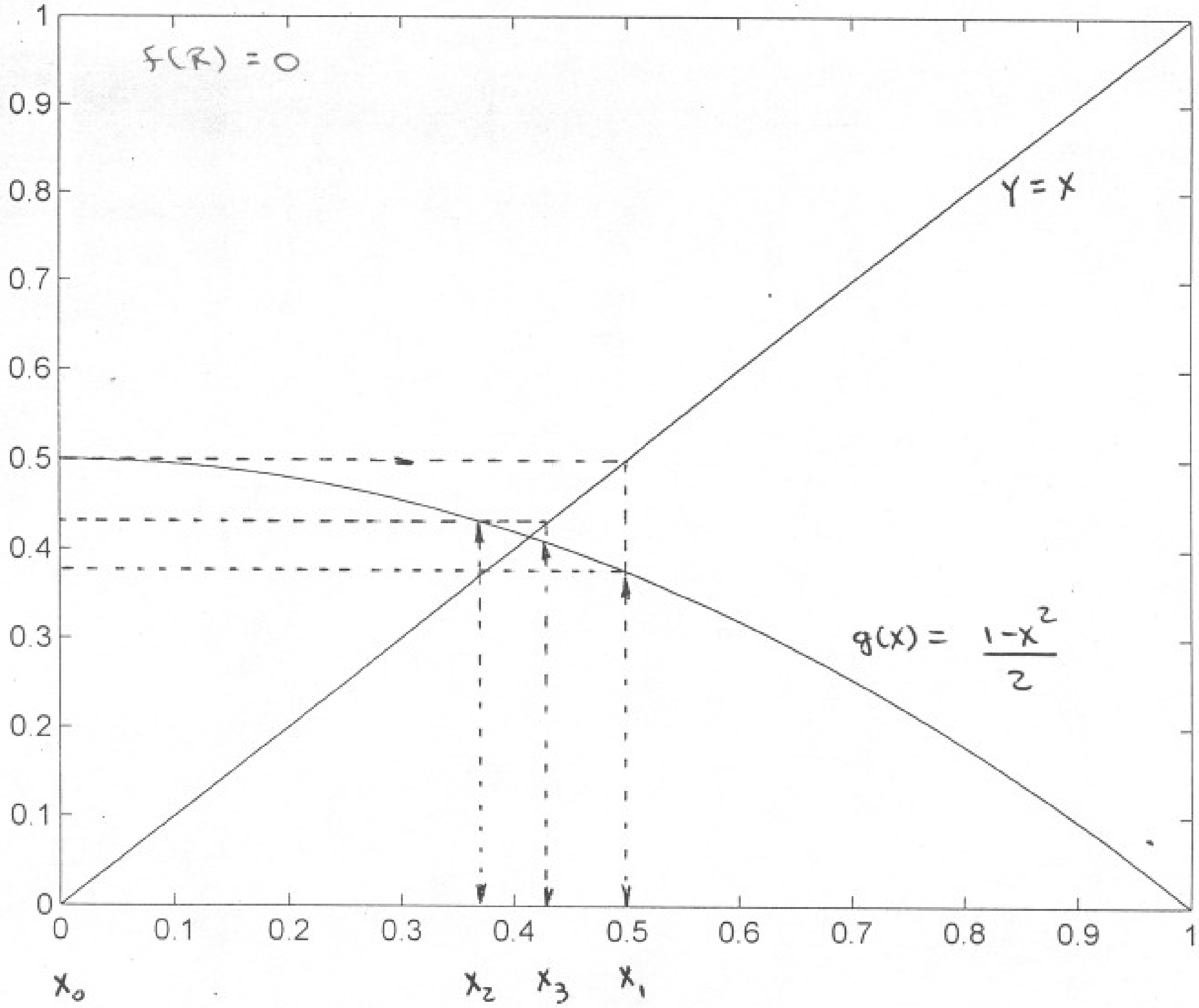
$f(x) = x^2 + 2x - 1$

At the root where $x = R$, $Y = X$ and $Y = g(x) = \frac{1-x^2}{2}$ are equal

$\Rightarrow \quad x = g(x)\big|_{x=R}$

$R = \frac{1-R^2}{2}$

$R^2 + 2R - 1 = 0$



Convergence

Let $E_{T,i} = R - x_i$ True error after the i^{th} iteration

It can be shown that

$$|E_{T,i+1}| = |g'(\xi)|\,|E_{T,i}| \quad \text{where } \xi \text{ is unknown}$$

Suppose the function $g(x)$ has a first derivative which satisfies $|g'(x)| < 1$ for $-\infty < x < \infty$

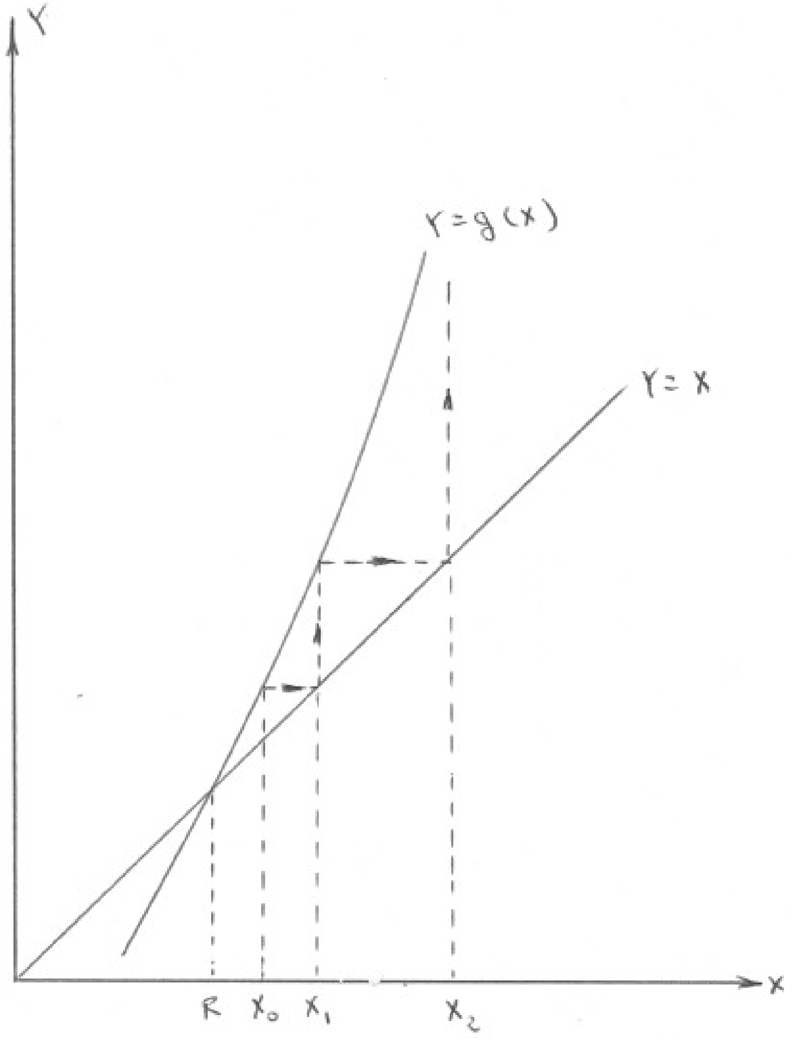
$$\Rightarrow \quad |E_{T,i+1}| < |E_{T,i}|$$

Therefore the true error decreases in magnitude with each iteration. Furthermore, the magnitude of the true error is roughly proportional to the magnitude of the previous true error. The constant of proportionality is $|g'(\xi)|$ which is less than 1.

Summarizing, $|g'(x)| < 1$ for $-\infty < x < \infty$ is a <u>sufficient</u> condition for convergence of the Simple Fixed Point Iteration Method. Using results from the previous example,

i	x_i	$E_{T,i} = R - x_i$	$\frac{\lvert E_{T,i}\rvert}{\lvert E_{T,i-1}\rvert}$
0	0	0.4142136	—
1	0.5	-0.0857864	0.207
2	0.375	0.0392136	0.457
3	0.4296875	-0.0154739	0.395
4	0.4076843	0.0065293	0.422
5	0.4168968	-0.0026832	0.411
6	0.4130986	0.00115	0.416

The following graph illustrates the case where the Simple Fixed Point Iteration Method fails to converge.



see Fig 6.3 on P 105 for a diagram of all 4 possible cases (2 convergent and 2 divergent)

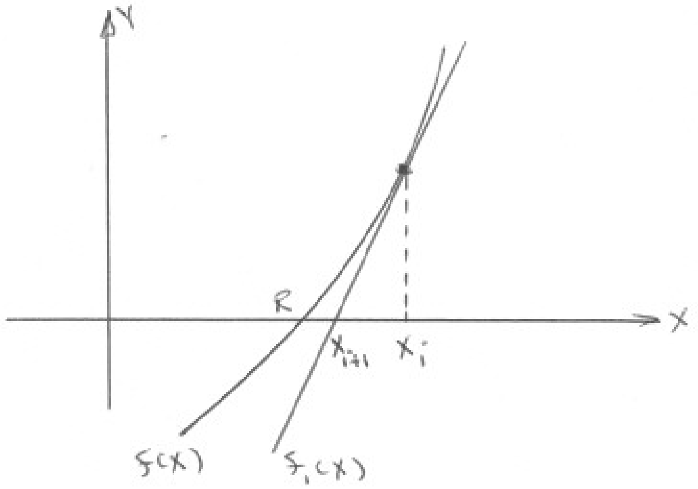
The Newton Raphson Method

Solving for the root(s) of $f(x)=0$ using an open method can be done by using a first order truncated Taylor Series Expansion.

Suppose x_i is the result of the i^{th} iteration aimed at estimating the root R of $f(x)=0$. Then the linear function

$$y = f_1(x) = f(x_i) + f'(x_i)(x - x_i)$$

is tangent to $y = f(x)$ at $x = x_i$.



Setting $y=0$, solving for x (and calling it x_{i+1})

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad , \quad i = 0, 1, 2, 3, ----$$

The above is called the Newton-Raphson Algorithm

Convergence of the Newton Raphson Algorithm

The Taylor Series of $f(x)$ about the point x_i is

$$f(x) = f(x_i) + f'(x_i)(x - x_i) + \frac{f''(\xi)}{2!}(x - x_i)^2 \quad , \quad x_i < \xi < x$$

Evaluating the Taylor Series Expansion at $x = R$,

$$f(R) = f(x_i) + f'(x_i)(R - x_i) + \frac{f''(\xi)}{2!}(R - x_i)^2 \quad , \quad x_i < \xi < R$$

Also, $\underline{0 \;=\; f(x_i) + f'(x_i)(x_{i+1} - x_i)}$ from the N-R Algorithm

$$\Rightarrow f(R) - 0 = f'(x_i)(R - x_{i+1}) + \frac{f''(\xi)}{2!}(R - x_i)^2$$

$$0 = f'(x_i)\, E_{T,i+1} + \frac{f''(\xi)}{2}(E_{T,i})^2$$

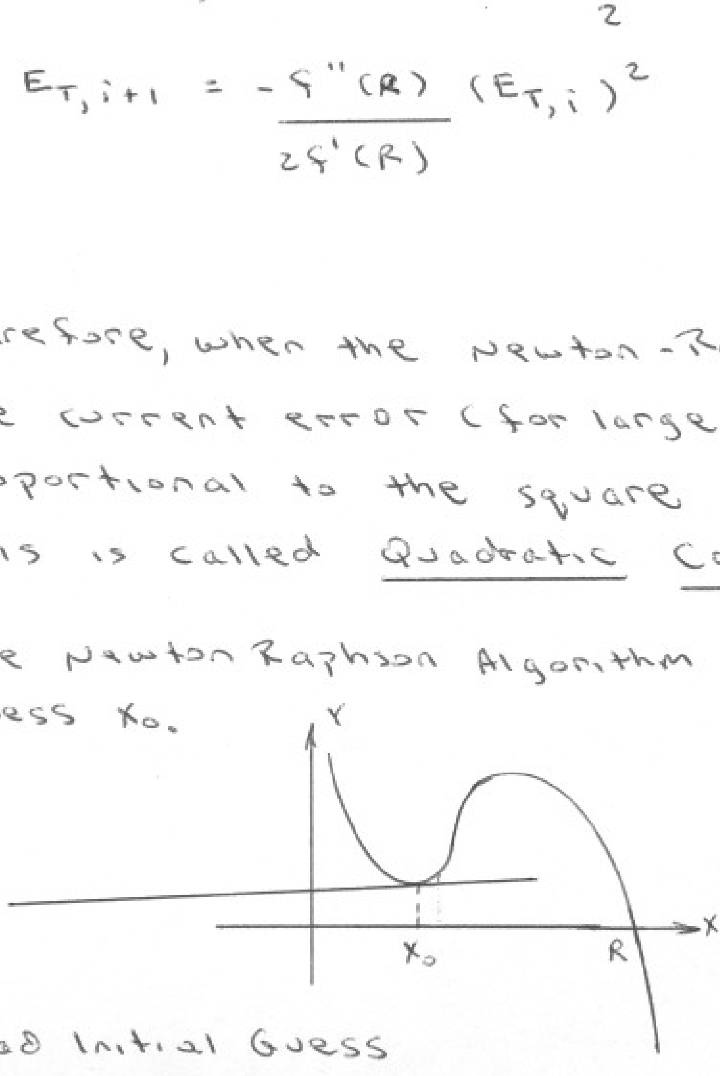
$$E_{T,i+1} = \frac{-f''(R)}{2f'(R)}(E_{T,i})^2$$

holds for i large assuming convergence

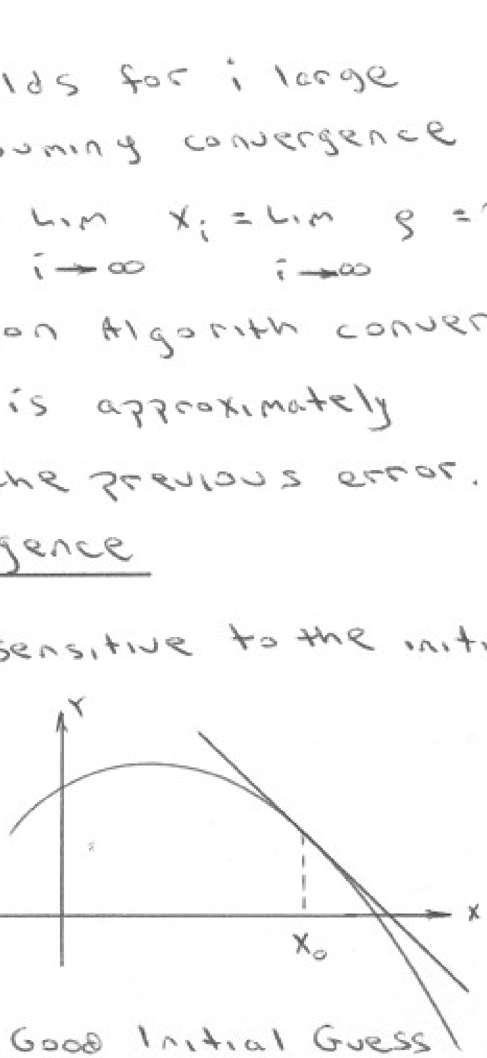
i.e. $\lim_{i \to \infty} x_i = \lim_{i \to \infty} \xi = R$

Therefore, when the Newton-Raphson Algorith converges, the current error (for large i) is approximately proportional to the square of the previous error. This is called <u>Quadratic</u> <u>Convergence</u>

The Newton Raphson Algorithm is sensitive to the initial guess x_0.



Bad Initial Guess



Good Initial Guess

ex. $f(x) = x^2 - 1$

$f'(x) = 2x$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{(x_i^2 - 1)}{2x_i}$$

$$= \frac{x_i^2 + 1}{2x_i}$$

i	x_i	$E_{T,i} = 1 - x_i$	$\left\lvert \frac{E_{T,i}}{(E_{T,i-1})^2} \right\rvert$
0	0.1	0.9	—
1	5.05	-4.05	5.0
2	2.624009901	-1.624009901	0.099009901
3	1.502553012	0.502553012	0.190548062
4	1.084043467	0.084043467	0.3327666960
5	1.003257851	0.003257851	0.461236107
6	1.000005290	0.00005290	0.498417447
7	1.000000000	0.000000000	

the last column is approaching

$$\frac{f''(R)}{2f'(R)} = \frac{2}{2[2(1)]} = 0.5$$