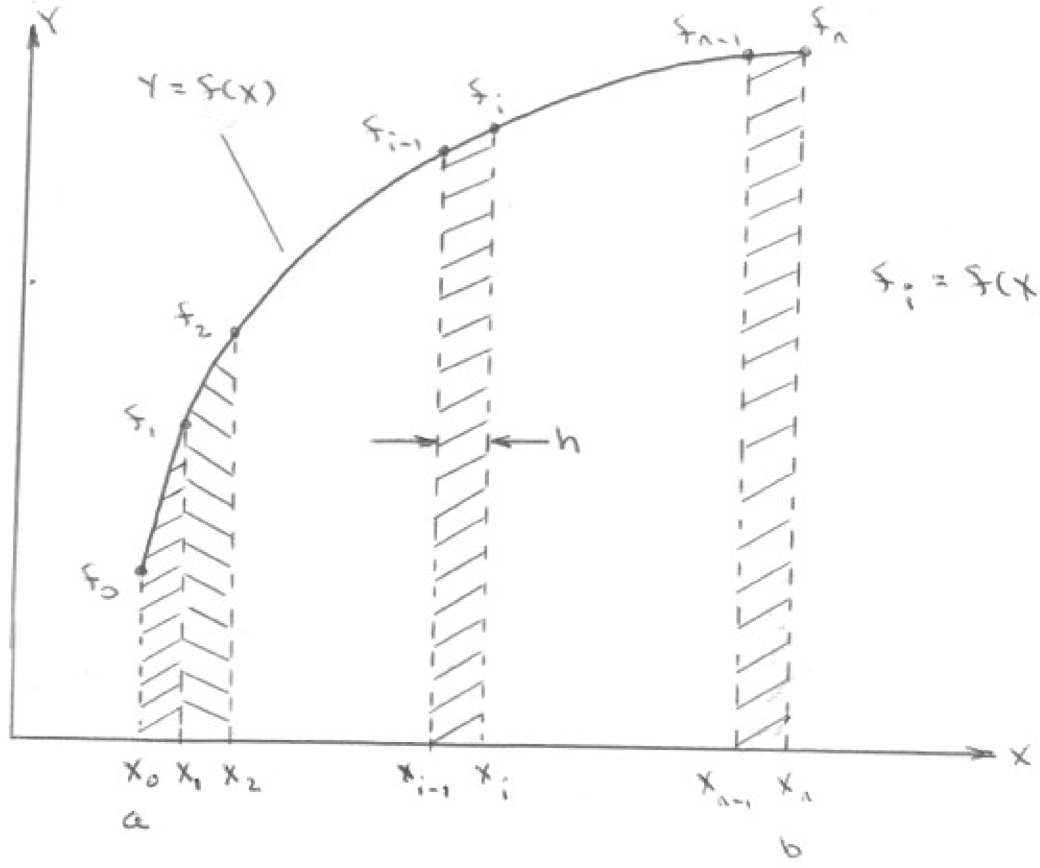
NUMERICAL INTEGRATION

Problem: Evaluate the definite integral

$$I = \int_a^b f(x)\,dx$$

where $f(x)$ is a continuous function of x for $a \le x \le b$

Trapezoidal Rule



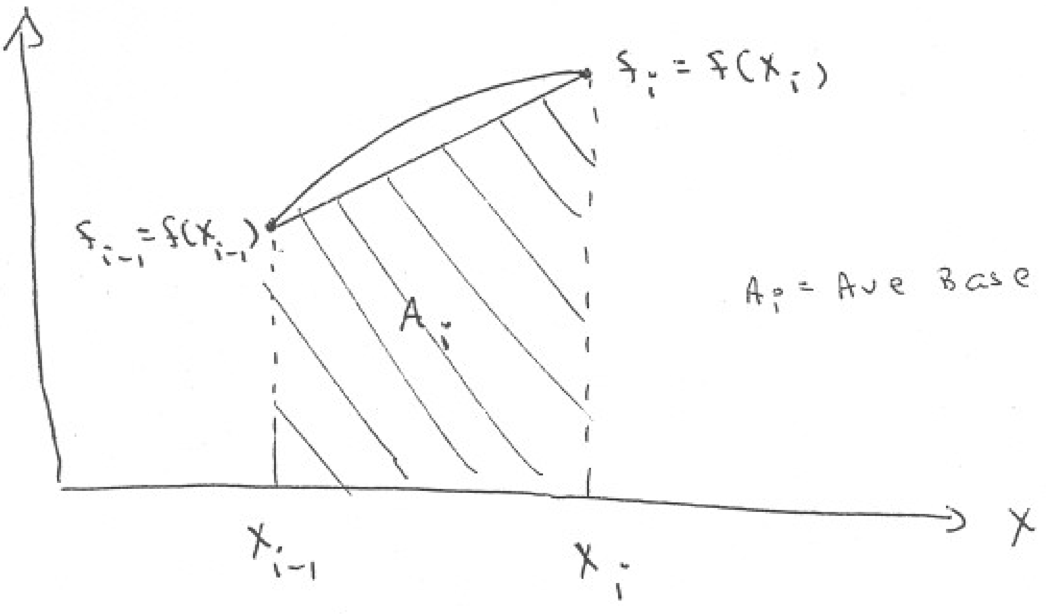
$f_i = f(x_i)$

Let $h = \dfrac{b-a}{n}$ & $x_i = a + ih$, $(i = 0, 1, 2, ---, n)$

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + ---+ \int_{x_{i-1}}^{x_i} f(x)\,dx + ---+ \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= I_1 + I_2 + ------ + I_i + ---- + I_n$$

Expanding the interval from x_{i-1} to x_i



A_i = Ave Base × Height

$$A_i = \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right](x_i - x_{i-1})$$

$$= \frac{h}{2}(f_{i-1} + f_i)$$

For small h, $A_i \simeq I_i \quad (i = 1, 2, \ldots, n)$

$$\Rightarrow \quad I \simeq A_1 + A_2 + \ldots + A_i + \ldots + A_n$$

$$\simeq \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \ldots + \frac{h}{2}(f_{i-1} + f_i) + \ldots + \frac{h}{2}(f_{n-1} + f_n)$$

$$\simeq \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n)$$ TRAPEZOIDAL RULE

or $$\simeq \frac{h}{2}\left[f_0 + 2\sum_{i=1}^{n-1} f_i + f_n\right]$$

or $$\simeq h\left[\left(\frac{f_0 + f_n}{2}\right) + \sum_{i=1}^{n-1} f_i\right]$$

$$I \approx \frac{(b-a)}{n}\left[\frac{(f_0+f_n)}{2} + \sum_{i=1}^{n-1} f_i\right]$$

$$I \approx (b-a)\left[\frac{f_0}{2n} + \frac{1}{n}\sum_{i=1}^{n-1} f_i + \frac{f_n}{2n}\right]$$

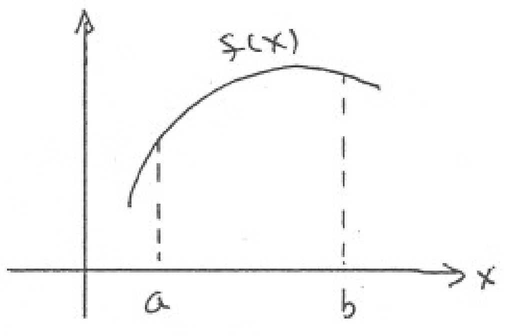
Newton - Cotes closed formula for Trapezoidal Integration

The quantity in brackets represents a weighted average of the function values $\{f_0, f_1, \ldots, f_{n-1}, f_n\}$.

Function Value	Weight
f_0	$w_0 = 1/2n$
f_1	$w_1 = 1/n$
$\vdots$	$\vdots$
f_{n-1}	$w_{n-1} = 1/n$
f_n	$w_n = 1/2n$

$$I \approx (b-a)\sum_{i=0}^{n} w_i f_i$$

From the Mean Value theorem in Calculus, the



average value of a function $f(x)$ over the interval (a,b) is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

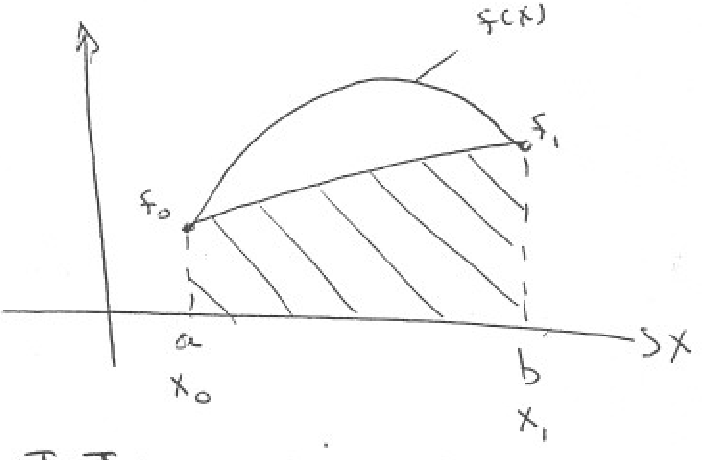
$$\Rightarrow \int_a^b f(x)\,dx = (b-a)\,\bar{f}$$

Thus, the bracketed term in the Newton-Cotes formula is an estimate of the average value of $f(x)$ over the interval (a,b).

Truncation Error in the Trapezoidal Rule

Single Application of Trapezoidal Rule

Let $I = \int_a^b f(x)\,dx$



& $I_1 = (b-a)\left[\frac{f_0+f_1}{2}\right]$

The local truncation error $I - I_1$ is given by

$$I - I_1 = -\frac{1}{12} \cdot f''(\xi)(b-a)^3 \quad \text{where} \quad a < \xi < b$$

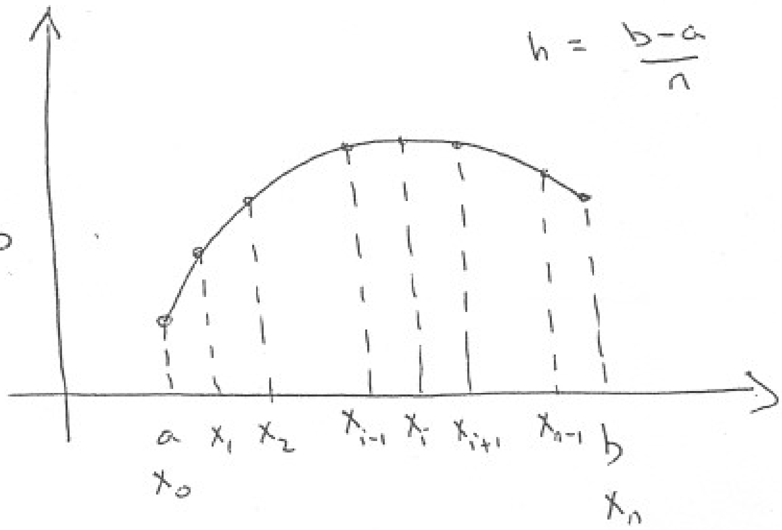
$$I - I_1 = -\frac{1}{12} f''(\xi)\, h^3 = O(h^3) \qquad \left(h = \frac{b-a}{1} = b-a\right)$$

Hence, if $f(x)$ is not linear, $I - I_1 \neq 0$ (in general)

Multiple Application of Trapezoidal Rule

In this case, the global truncation error is given by

$h = \frac{b-a}{n}$



$$I - I_n = \frac{-1}{12} f''(\xi) \frac{(b-a)^3}{n^2}, \quad a < \xi < b$$

$$= \frac{-1}{12} f''(\xi)(b-a)\, h^2$$

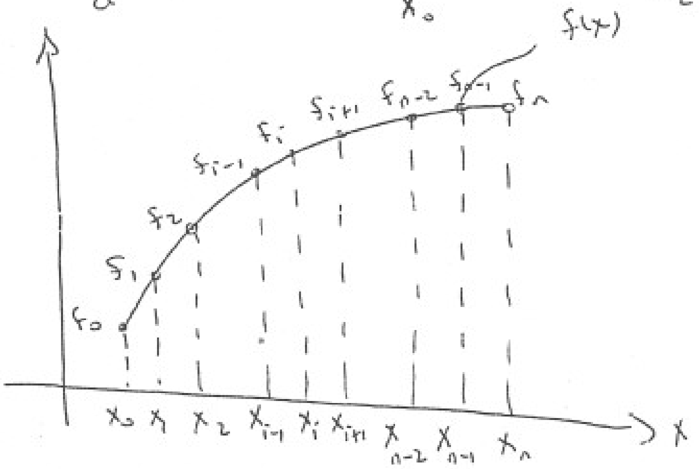
$$= O(h^2)$$

Simpson's Rules

Suppose the interval (a,b) is divided into an even number of intervals n of width h,

$$h = \frac{b-a}{n}, \quad x_i = a + ih, \quad i = 0, 1, 2, \dots, n$$

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots + \int_{x_{i-1}}^{x_{i+1}} f(x)\,dx + \dots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$



$$\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx \approx \int_{x_{i-1}}^{x_{i+1}} f_{2,i}(x)\,dx$$

where $f_{2,i}(x)$ is a 2nd order interpolating polynomial thru $(x_{i-1}, f_{i-1}), (x_i, f_i), (x_{i+1}, f_{i+1})$

For example, the Lagrange 2nd order interpolating polynomial is

$$f_{2,i}(x) = \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} f_{i-1}$$

$$+ \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i - x_{i-1})(x_i - x_{i+1})} f_i$$

$$+ \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} f_{i+1}$$

It can be shown that $\displaystyle\int_{x_{i-1}}^{x_{i+1}} f_{2,i}(x)\,dx = \frac{h}{3}\left(f_{i-1} + 4f_i + f_{i+1}\right)$

$$I = \frac{h}{3}\left(f_0 + 4f_1 + f_2\right) + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right) + \cdots$$

$$\cdots + \frac{h}{3}\left(f_{i-1} + 4f_i + f_{i+1}\right) + \cdots + \frac{h}{3}\left(f_{n-2} + 4f_{n-1} + f_n\right)$$

$$I = \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n\right)$$

This is Simpsons 1/3 Rule.

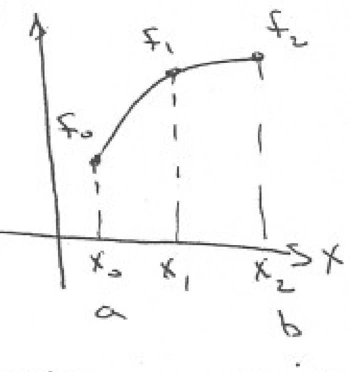
Using the Newton-Coates form,

$$I = (b-a)\frac{\left[f_0 + 4\sum_{i=1,3,5}^{n-1} f_i + 2\sum_{i=2,4,6}^{n-2} f_i + f_n\right]}{3n}$$

Function Value	weight
f_0	$1/3n$
$f_1, f_3, f_5, \ldots, f_{n-1}$	$4/3n$
$f_2, f_4, f_6, \ldots, f_{n-2}$	$2/3n$
f_n	$1/3n$

Local Truncation Error is

$$I - I_2 = -\frac{1}{90} f^4(\xi) h^5$$



The global truncation error is

$$I - I_n = -\frac{1}{180} f^4(\xi)\frac{(b-a)^5}{n^4}$$

$$= -\frac{1}{180} f^4(\xi)(b-a)h^4$$

$$= O(h^4)$$

Note that if $f(x)$ is a cubic then $f^4(\xi) = 0$, $a < x < b$ and $I - I_n = 0$

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SHOW ALL WORK!

Problem 3 (25 pts)

Evaluate the integral $\int_0^{\pi}(4+2\sin x)\,dx$

a) Analytically
b) By trapezoidal integration using 8 intervals
c) By Simpson's 1/3 formula using 4 intervals

Work Area

a) $\int_0^{\pi}(4+2\sin x)\,dx = 4x - 2\cos x\Big|_0^{\pi} = 4\pi - 2\cos\pi + 2\cos 0 = 16.56637$

b) (n=8)

i	x_i	$f(x_i)$
0	0	4.00000
1	$\pi/8$	4.76537
2	$\pi/4$	5.41421
3	$3\pi/8$	5.84776
4	$\pi/2$	6.00000
5	$5\pi/8$	5.84776
6	$3\pi/4$	5.41421
7	$7\pi/8$	4.76537
8	π	4.00000

$I_T \doteq \frac{b-a}{n}\left[\frac{f(x_0)}{2} + f(x_1) + \cdots + f(x_7) + \frac{f(x_8)}{2}\right]$

$= 16.51483$

c) (n=4)

i	x_i	$f(x_i)$
0	0	4.00000
1	$\pi/4$	5.41421
2	$\pi/2$	6.00000
3	$3\pi/4$	5.41421
4	π	4.00000

$I_S = \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$

$= 16.57548$

Ans. a) 16.56637 b) 16.51483 c) 16.57548

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SHOW ALL WORK!

Problem 3 (30 pts)

Approximate the integral $I = \int_0^1 \sin\frac{\pi x}{2}\,dx$

a) By trapezoidal integration using 8 intervals

b) By Simpson's 1/3 formula using 8 intervals

You may use the remaining columns in the table to help solve the problem. Round all answers to 4 places after the decimal point.

i	x_i	$f_i = f(x_i)$
0	0	0
1	0.125	0.1951
2	0.25	0.3827
3	0.375	0.5556
4	0.5	0.7071
5	0.625	0.8315
6	0.75	0.9239
7	0.875	0.9808
8	1	1

a) $h = \frac{b-a}{n} = \frac{1-0}{8} = 0.125$

$f_i = f(x_i) = \sin\frac{\pi x_i}{2}$

$I_T = h\left[\frac{f_0+f_8}{2} + f_1+f_2+f_3+f_4+f_5+f_6+f_7\right] = \underline{0.6346}$

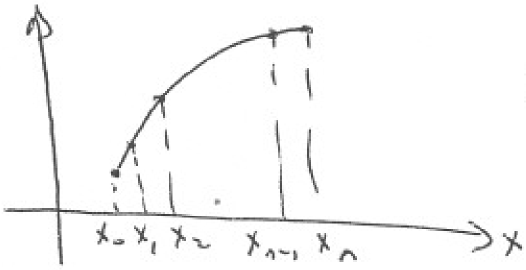
$I_S = \frac{h}{3}\left[f_0+4(f_1+f_3+f_5+f_7)+2(f_2+f_4+f_6)+f_8\right] = \underline{0.6366}$

Romberg Integration

Richardson's Extrapolation Formula

The Trapezoidal Rule can be used twice with different interval step sizes and the results used to compute a new more accurate estimate of $I = \int_a^b f(x)\,dx$.

Suppose the interval (a, b) is divided into n equal subintervals, $h = \frac{b-a}{n}$. Then



$$I = I(h) + E(h)$$

where $I(h)$ is the approximate value of I based on n subintervals of size h

$E(h)$ is the truncation error

Suppose the trapezoidal rule is applied twice, once with step size $h_1 = \frac{b-a}{n_1}$ and once with $h_2 = \frac{b-a}{n_2}$.

$$\Rightarrow \quad I = I(h_1) + E(h_1)$$

$$I = I(h_2) + E(h_2)$$

$$\Rightarrow \quad I(h_1) + E(h_1) = I(h_2) + E(h_2)$$

Recall that the global truncation error of the Trapezoidal Rule with step size h is $E(h) = -\frac{1}{12}(b-a) f''(\xi) h^2$, $a \le \xi \le b$

$$\frac{E(h_1)}{E(h_2)} = \frac{-1/12\,(b-a) f''(\xi_1) h_1^2}{-1/12\,(b-a) f''(\xi_2) h_2^2}$$

Assuming $f''(x) = \text{constant}$ for $a \le x \le b$

$$\Rightarrow \frac{E(h_1)}{E(h_2)} = \frac{h_1^2}{h_2^2}$$

From which we get, $E(h_1) = \frac{h_1^2}{h_2^2} E(h_2)$

$$\Rightarrow I(h_1) + \left(\frac{h_1}{h_2}\right)^2 E(h_2) = I(h_2) + E(h_2)$$

$$\Rightarrow I(h_1) - I(h_2) = \left[1 - \left(\frac{h_1}{h_2}\right)^2\right] E(h_2)$$

$$\Rightarrow E(h_2) = \frac{I(h_1) - I(h_2)}{1 - \left(\frac{h_1}{h_2}\right)^2}$$

The above is an estimate of the truncation error in $I(h_2)$ based on the $I(h_1)$, h_1, $I(h_2)$ & h_2.

Substituting back into $I = I(h_2) + E(h_2)$ yields

$$I = I(h_2) + \frac{I(h_1) - I(h_2)}{1 - \left(\frac{h_1}{h_2}\right)^2}$$

This estimate of I is in error (recall underlying assumption). However the error is $O(h^4)$, even though the errors in $I(h_1)$ & $I(h_2)$ are both $O(h^2)$.

<u>Special</u> <u>Case</u>: $h_2 = h_1/2$

$$\Rightarrow I = I(h_2) + \frac{I(h_1) - I(h_2)}{1 - 2^2} = \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1)$$

ex. $I = \int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)\,dx = 1.64053334$

Suppose Trapezoidal integration is used with 1, 2 and 4 subintervals. The following table results.

subintervals	h	I(h)	e_T (%)
1	0.8	0.1728	89.5
2	0.4	1.0688	34.9
4	0.2	1.4848	9.5

combining the 1st and 2nd estimate,

$h_1 = 0.8$, $I(h_1) = 0.1728$

$h_2 = 0.4$, $I(h_2) = 1.0688$

$I \approx \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1)$

$= \frac{4}{3}(1.0688) - \frac{1}{3}(0.1728)$

$= 1.36746667$

$e_T = \left(\frac{1.64053334 - 1.36746667}{1.64053334}\right) \times 100 = 16.6\%$

combining the 2nd and 3rd estimate

$h_2 = 0.4$, $I(h_2) = 1.0688$

$h_3 = 0.2$, $I(h_3) = 1.4848$

$I = \frac{4}{3} I(h_3) - \frac{1}{3} I(h_2)$

$= \frac{4}{3}(1.4848) - \frac{1}{3}(1.0688)$

$= 1.62346667$, $e_T = \left(\frac{1.64053334 - 1.62346667}{1.64053334}\right) \times 100 = 1.04\%$

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Problem 5 (30 pts)

Consider the following definite integral: $I = \int_0^1 \frac{dx}{1+x}$

a) Use Trapezoidal integration with 4 sub-intervals to approximate I.
b) Use Trapezoidal integration with 8 sub-intervals to approximate I.
c) Use the results of the first two parts to find an improved approximation of I.
d) Use Simpson's 1/3 Rule with 8 sub-intervals to approximate I.
e) Find the global truncation error in Part 4.

Note: $I = \Big[\ln(1+x) \Big]_0^1 = \ln 2$

Fill in the table below to help with the calculations. All intermediate and final results should be rounded to five places after the decimal point.

i	x_i	$f_i = f(x_i)$
0	0.000	
1	0.125	
2	0.250	
3	0.375	
4	0.500	
5	0.625	
6	0.750	
7	0.875	
8	1.000	

Ans. I_4 = ____________ (Trapezoidal, n=4)
I_8 = ____________ (Trapezoidal, n=8)
$I_{4/8}$ = ____________ (Repeated Trapezoidal, n=4, m=8)
I_8 = ____________ (Simpson's 1/3 Rule, n=8)

Global Truncation Error = ____________ (Simpson's 1/3 Rule)

Work Area

i	x_i	$f_i = f(x_i) = \dfrac{1}{1+x_i}$
0	0	1.00000
1	0.125	0.88889
2	0.250	0.80000
3	0.375	0.72727
4	0.500	0.66667
5	0.625	0.61538
6	0.750	0.57143
7	0.875	0.53333
8	1	0.50000

a) $I_4 = h\left[\dfrac{f_0+f_4}{2} + f_1 + f_2 + f_3\right]$ where $h = 0.25$

$= 0.25\left[\dfrac{1+0.5}{2} + 0.8 + 0.66667 + 0.57143\right]$

$= 0.69702$

b) $I_8 = h\left[\dfrac{f_0+f_8}{2} + f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7\right]$ where $h = 0.125$

$= 0.125\left[\dfrac{1+0.5}{2} + 0.88889 + 0.80000 + \dots + 0.53333\right]$

$= 0.69412$

c) $I_{4/8} = I_4 + \dfrac{I_4 - I_8}{\left(\frac{0.125}{0.250}\right)^2 - 1} = 0.69702 + \dfrac{0.69702 - 0.69412}{(1/2)^2 - 1}$

$= 0.69315$

d) $I_8 = \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + \dots + 2f_6 + 4f_7 + f_8\right)$

$= \frac{h}{3}\left[f_0 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8\right]$

$= \frac{0.125}{3}\left[1 + 4(0.88889 + \dots + 0.53333) + 2(0.8 + \dots + 0.57143) + 0.5\right]$

$= 0.69315$

e) Global Truncation Error $= I - I_8$

$= \ln 2 - 0.69315$

$= 0.69315 - 0.69315$

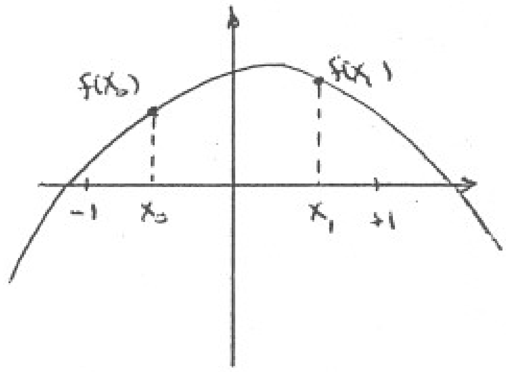
$= 0$

Gauss Quadrature

Derivation of Two Point Gauss-Legendre Formula

Suppose we wish to approximate $I = \int_{-1}^{1} f(x)\,dx$ by using a linear combination of functional values,

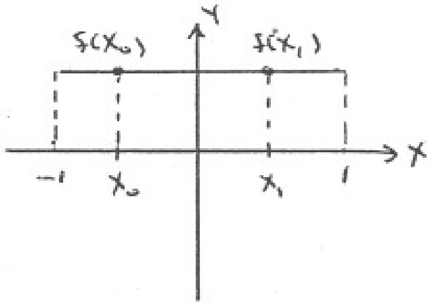
i.e. $I \approx c_0 f(x_0) + c_1 f(x_1)$



where c_0, c_1 and x_0, x_1 are to be determined.

Four equations are req'd to determine the unknowns.

1) if $f(x) = 1$, $-1 \le x \le 1$ the approximation is exact

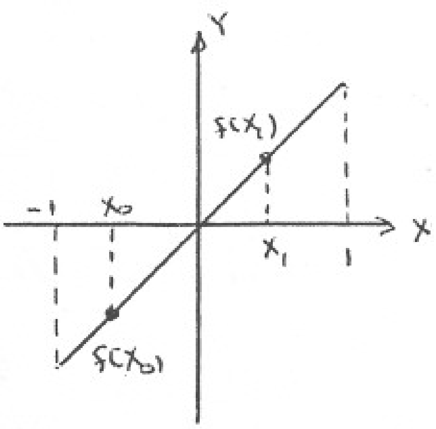


$$I = \int_{-1}^{1} 1\,dx = 2$$

$$c_0 f(x_0) + c_1 f(x_1) = c_0(1) + c_1(1)$$

$$\Rightarrow \quad c_0 + c_1 = 2 \qquad \text{EQ } 1$$

2) if $f(x) = x$, $-1 \le x \le 1$ the approximation is exact



$$I = \int_{-1}^{1} x\,dx = 0$$

$$c\, f(x_0) + c_1 f(x_1) = c_0(x_0) + c_1(x_1)$$

$$\Rightarrow \quad c_0 x_0 + c_1 x_1 = 0 \qquad \text{EQ } 2$$

3) if $f(x) = x^2$, $-1 \le x \le 1$ the approximation is exact

$$I = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \qquad c_0 f(x_0) + c_1 f(x_1) = c_0 x_0^2 + c_1 x_1^2$$

$$\Rightarrow \quad c_0 x_0^2 + c_1 x_1^2 = \frac{2}{3} \qquad \text{EQ 3}$$

4) if $f(x) = x^3$, $-1 \le x \le 1$ the approximation is exact

$$I = \int_{-1}^{1} x^3 \, dx = 0 \qquad c_0 f(x_0) + c_1 f(x_1) = c_0 x_0^3 + c_1 x_1^3$$

$$\Rightarrow \quad c_0 x_0^3 + c_1 x_1^3 = 0 \qquad \text{EQ 4}$$

The solution to EQ's 1-4 is $c_0 = c_1 = 1$

$$x_0 = \frac{-1}{\sqrt{3}}, \quad x_1 = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad I \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \qquad \text{Third order accurate}$$

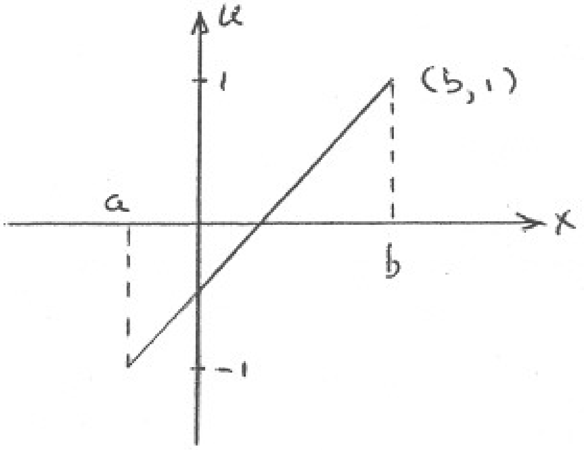
For the general case with integration limits a & b, the variable of integration x must be transformed into a new variable u which varies from -1 to 1 when x goes from a to b.

$$u - u_0 = m(x - x_0)$$

$$\Rightarrow u - 1 = \frac{2}{b-a}(x - b)$$

$$\Rightarrow x = \frac{(b-a)u + (a+b)}{2}$$

$$\Rightarrow dx = \left(\frac{b-a}{2}\right) du$$



$$\Rightarrow \int_a^b f(x)\,dx = \int_{-1}^{1} f\left[\frac{(b-a)u+(a+b)}{2}\right]\left(\frac{b-a}{2}\right)du$$

ex. $I = \int_0^4 x e^{2x}\,dx \qquad (a=0,\ b=4)$

$$\Rightarrow x = \frac{(4-0)u+(0+4)}{2} = 2(u+1)$$

$$dx = \left(\frac{4-0}{2}\right)du = 2\,du$$

$$I = \int_{-1}^{1} 2(u+1)\,e^{2\cdot 2(u+1)}\,2\,du$$

$$= \int_{-1}^{1} 4(u+1)\,e^{4(u+1)}\,du$$

$$= \int_{-1}^{1} f(u)\,du \quad \text{where } f(u) = 4(u+1)e^{4(u+1)}$$

$$\Rightarrow I \approx \left[f\left(\frac{-1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$\simeq 4\left\{\left(\frac{-1}{\sqrt{3}}\right)+1\right\}e^{4\left\{\left(\frac{-1}{\sqrt{3}}\right)+1\right\}} + 4\left\{\left(\frac{1}{\sqrt{3}}\right)+1\right\}e^{4\left\{\left(\frac{1}{\sqrt{3}}\right)+1\right\}}$$

$$\simeq 3477.543936$$

Exact Value, $I = 5216.926477$

$$e_T = \left(\frac{I - 3477.543936}{I}\right)\times 100 = 33.3\ \%$$

Higher Point Formulas

The general form of the Gauss Legendre Formula is

$$I \simeq c_0 f(x_0) + c_1 f(x_1) + \cdots + c_{n-1} f(x_{n-1})$$

The values for c_i, x_i $(i = 0, 1, \ldots, n-1)$ are obtained in a similar way to the Two Point Formula.

i.e. with 3 pts $(n=2)$

c_0, c_1, c_2 & x_0, x_1, x_2 are chosen so that $I = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$ is exact for polynomial integrands up thru $f(x) = x^5$. The results are

$c_0 = 0.555555556$ $\qquad x_0 = -0.774596669$

$c_1 = 0.888888889$ $\qquad x_1 = 0$

$c_2 = 0.555555556$ $\qquad x_2 = 0.774596669$

The estimate is 5^{th} order accurate

ex. Repeat previous problem with 3 pt formula

$$I \simeq 0.555555556 f(-0.774596669)$$
$$+ 0.888888889 f(0)$$
$$+ 0.555555556 f(0.774596669)$$

where $f(x) = 4(x+1)e^{4(x+1)}$

$\Rightarrow$ $I \simeq 5197.54375$

$$e_T = \left(\frac{5216.926477 - 5197.54375}{5216.926477}\right) \times 100 = 0.37\%$$

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SHOW ALL WORK!

Problem 1 (35 pts)

Consider the definite integral

$$I=\int_a^b f(x)\,dx \quad \text{where } a=0\text{ , } b=1 \text{ and } f(x)=e^{-x^2},$$

A) Use Simpson's Rule with 8 intervals to approximate I. Fill in the table below with $f(x_i)$ rounded to 4 places after the decimal point. Express your answer to 4 places after the decimal point.

i	x_i	$f_i=f(x_i)$
0	0.0000	1.0000
1	0.1250	0.9845
2	0.2500	0.9394
3	0.3750	0.8688
4	0.5000	0.7788
5	0.6250	0.6766
6	0.7500	0.5698
7	0.8750	0.4650
8	1.0000	0.3679

$$I=\frac{1/8}{3}[f_0+4(f_1+f_3+f_5+f_7)+2(f_2+f_4+f_6)+f_8]$$

$$=0.7468$$

B) Use the Gauss Quadrature two point formula to approximate I. Express your answer to 4 places after the decimal point.

$$I = \int_0^1 e^{-x^2}\,dx \quad , \quad a=0, \quad b=1, \quad f(x) = e^{-x^2}$$

$$x = \frac{(b-a)u + (a+b)}{2} = \frac{u+1}{2}$$

$$dx = \left(\frac{b-a}{2}\right) du = \frac{du}{2}$$

$$I = \int_{-1}^{1} f\left(\frac{u+1}{2}\right) \frac{du}{2}$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-\left(\frac{u+1}{2}\right)^2} du$$

$$= \frac{1}{2} \int_{-1}^{1} F(u)\,du \quad , \quad \text{where } F(u) = e^{-\left(\frac{u+1}{2}\right)^2}$$

Using the Gauss Quadrature two point formula,

$$I \approx \frac{1}{2}\left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$\approx 0.7466$$

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Note: There are only 3 problems. You must do all 3 problems!

SHOW ALL WORK!

Problem 1 35 pts

Consider the definite integral

$$I = \int_a^b f(x)\,dx \quad \text{where } a = 3.1\text{ , } b = 3.9 \text{ and } f(x) = \frac{1}{x},$$

1. Use Simpson's Rule with 8 intervals to approximate I. Fill in the table below with $f(x_i)$ rounded to 8 places after the decimal point. Express your answer to 8 places after the decimal point.

i	x_i	$f(x_i)$
0	3.1	0.32258065
1	3.2	0.31250000
2	3.3	0.30303030
3	3.4	0.29411765
4	3.5	0.28571429
5	3.6	0.27777777
6	3.7	0.27027027
7	3.8	0.26315789
8	3.9	0.25641025

2. Use the Gauss Quadrature two point formula to approximate I.

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1. Simpsons Rule: $h = \dfrac{b-a}{n} = \dfrac{3.9-3.1}{8} = 0.1$

$$I = \frac{h}{3}\{f(x_0) + 4[f(x_1) + f(x_3) + f(x_5) + f(x_7)] + 2[f(x_2) + f(x_4) + f(x_6)] + f(x_8)\}$$

$= 0.22957446$

$$I = \int_a^b f(x)\,dx = \int_{-1}^{1} F(t)\,dt$$

$a = 3.1$, $b = 3.9$ $\qquad f(x) = \frac{1}{x}$

$$x = \left(\frac{b-a}{2}\right)t + \frac{a+b}{2}$$

$$= 0.4t + 3.5$$

$$dx = 0.4\,dt$$

$$I = \int_{-1}^{1} \frac{1}{0.4t+} \; 0.4\,dt$$

$$F(t) = \frac{1}{0.4t + 3.5}$$

$$I = 0.4\left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= 0.4\left[\frac{1}{0.4\left(\frac{-1}{\sqrt{3}}\right)+3.5} + \frac{1}{0.4\left(\frac{1}{\sqrt{3}}\right)+3.5}\right]$$

$$= 0.4\,(0.30589834 + 0.26802896)$$

$$= 0.22957092$$