Systems of Linear Algebraic Equations

Given a set of m equations in n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + b_{21}x_2 + \cdots\cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m$$

a solution is any set of numbers $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ which satisfies the entire set of equations.

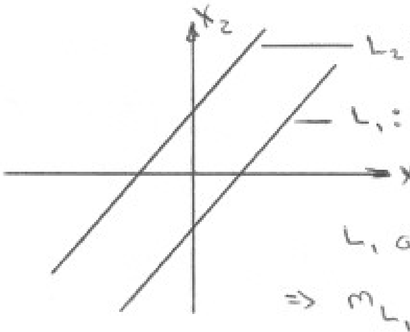
Suppose $m = n = 2$,

$$\Rightarrow \quad a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

There are 3 possible outcomes involving solutions

I. No Solution



$L_2: a_{21}x_1 + a_{22}x_2 = b_2$

$L_1: a_{11}x_1 + a_{12}x_2 = b_1$

L_1 and L_2 are parallel

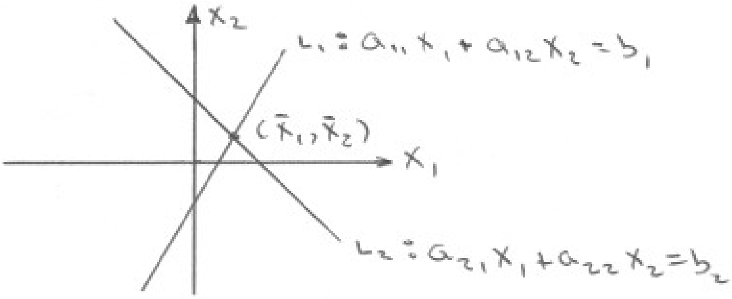
$$\Rightarrow m_{L_1} = m_{L_2}$$

$$\frac{-a_{11}}{a_{12}} = \frac{-a_{21}}{a_{22}}$$

$$-a_{11}a_{22} = -a_{12}a_{21}$$

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

II. One Solution

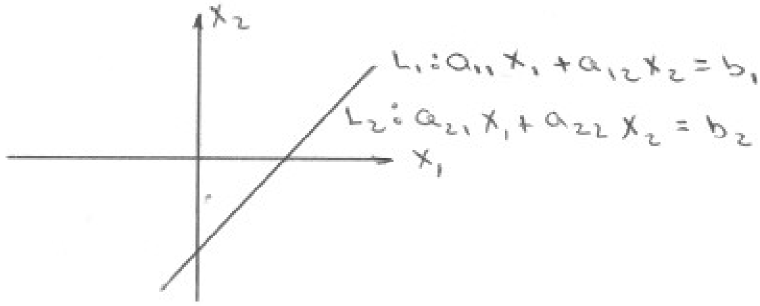


L_1 and L_2 intersect

$$\Rightarrow \quad m_{L_1} \neq m_{L_2}$$

$$a_{11}a_{22} - a_{12}a_{21} \neq 0$$

III. Infinite Solutions



L_1 and L_2 are the same line

$$m_{L_1} = m_{L_2}$$

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

The determinant of the left hand side coefficients is

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Therefore, the 2×2 system of equations has a unique solution only if $D \neq 0$. Otherwise when $D = 0$, there is either none or an infinite number of solutions.

The same conditions apply to an $n \times n$ system of equations i.e. A unique solution exists only when the determinant of the left hand side coefficients is nonzero.

ex.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 \qquad + 4x_3 &= -1 \end{aligned}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 0 & 4 \end{vmatrix} = 0$$

Each equation represents a plane in 3-dimensional space. Since $D = 0$, there is not a single point, i.e. just one, that is on all 3 planes. Therefore the 3x3 system of equations does not possess a unique solution.

Note, the right hand side constants are irrelevant when it comes to determining the existence of a unique solution. The question of whether there are no solutions (3x3 system is inconsistent) or infinite solutions (3x3 system is consistent) will be addressed later on.

ex.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + 2x_2 + 4x_3 &= -1 \end{aligned}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & 4 \end{vmatrix} \neq 0$$

There is a unique solution, i.e. a single point in 3-dimensional space where all 3 planes intersect. When a unique solution exists to an nxn system of equations, Cramer's Rule can be used to find it.

$$x_1 = \frac{\begin{vmatrix} 0 & 1 & 1 \\ -1 & -1 & 3 \\ -1 & 2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & 4 \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 3 & -1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & 4 \end{vmatrix}}, \quad x_3 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & -1 & -1 \\ 3 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & 4 \end{vmatrix}}$$

Introduction to Matrices

A matrix is a rectangular array of elements (usually numbers) represented by a symbol.

ex. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 4 \\ 7 & -2 \\ 6 & 0 \end{pmatrix}$

For an $m \times n$ matrix (m rows and n columns) A, the element located at the intersection of the i^{th} row and j^{th} column is designated a_{ij}. In the 2×3 matrix B

$b_{11} = 1$ $\quad b_{21} = 4$

$b_{12} = 2$ $\quad b_{22} = 5$

$b_{13} = 3$ $\quad b_{23} = 6$

Note: A matrix has no numerical value unlike a determinant (which is always square, i.e. $n \times n$)

$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \\ 5 & 6 & 3 \end{pmatrix}$ is a matrix

$|A| = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \\ 5 & 6 & 3 \end{vmatrix}$ is a determinant which can be evaluated to produce a numerical value.

An $m \times n$ matrix A reduces to a row vector when $m = 1$ and a column vector when $n = 1$.

$\underline{a} = (0 \ 2 \ 1 \ 3)$, row vector ($1 \times 4$ matrix)

$\underline{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, column vector ($3 \times 1$ matrix)

Vectors are represented by lower case letters that are underlined.

A matrix is <u>diagonal</u> if it is square, $m=n$, and the only non-zero elements are on the principal diagonal.

ex.
$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$a_{ij} = 0, \ i \neq j$ $\quad i,j = 1,2,3$

$b_{ij} = 0, \ i \neq j$ $\quad i,j = 1,2,3,4$

A matrix is an <u>Identity matrix</u> if it is a diagonal matrix and all the elements on the principal diagonal are 1.

ex.
$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I = (1), \quad I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A matrix is <u>upper triangular</u> if its a square matrix and all the elements below the main diagonal are zero.

i.e. Given A - $n \times n$ matrix

$$a_{ij} = 0, \quad i > j$$
$$i = 1, 2, \ldots, n$$
$$j = 1, 2, \ldots, n$$

ex
$$A = \begin{pmatrix} 2 & 3 & 0 & -1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Matrix Operations</u>

Given two $m \times n$ matrices, A & B

(1) They are equal if $a_{ij} = b_{ij}$ $\quad i = 1,2,3,\ldots, m$; $\ j = 1,2,3,\ldots, n$

$A = B$

(2) The sum is $C = A + B$ where $c_{ij} = a_{ij} + b_{ij}$

$i = 1, 2, 3, ---, m$

$j = 1, 2, 3, ----, n$

(3) The scalar multiple of A, i.e. λA is given by

$$\lambda A = \lambda \begin{pmatrix} a_{11} & ---- & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & --- & a_{mn} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & ---- & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & ----- & \lambda a_{mn} \end{pmatrix}$$

ex. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, $\lambda = -2$

$$\begin{aligned} A + \lambda B &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + (-2) \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -8 & -6 \\ -4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -7 & -4 \\ -1 & 2 \end{pmatrix} \end{aligned}$$

Matrix Multiplication

Given two matrices A and B, the product AB is defined only when the number of columns in A equals the number of rows in B.

Suppose A is $m \times p$
B is $p \times n$

$C = AB$ is an $m \times n$ matrix where

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}, \quad i = 1, 2, ----, m$$

$j = 1, 2, ---, n$

ex. $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 \\ 1 & -3 \\ 0 & 1 \end{pmatrix}$

3×3 $\qquad$ 3×2

$$C = AB = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}, \quad c_{ij} = \sum_{k=1}^{3} a_{ik} b_{kj} \qquad \begin{matrix} i = 1, 2, 3 \\ j = 1, 2 \end{matrix}$$

3×2

$$\Rightarrow c_{11} = \sum_{k=1}^{3} a_{1k} b_{k1} = a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31}$$
$$= 1(2) + 0(1) + 2(0)$$
$$= 2$$

$$c_{12} = \sum_{k=1}^{3} a_{1k} b_{k2} = a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32}$$
$$= 1(4) + 0(-3) + 2(1)$$
$$= 6$$

$\vdots$

$$c_{32} = \sum_{k=1}^{3} a_{3k} b_{k2} = a_{31} b_{12} + a_{32} b_{22} + a_{33} b_{32}$$
$$= 5(4) + 6(-3) + 0(1)$$
$$= 2$$

$$C = \begin{pmatrix} 2 & 6 \\ 5 & 19 \\ 16 & 2 \end{pmatrix}$$

Matrix multiplication is associative and distributive

$$\Rightarrow \quad ABC = A(BC) = (AB)C$$

$$A(B+C) = AB + AC$$

assuming all the above matrix products are defined.

Matrix multiplication is not commutative.

$AB \neq BA$ as a general rule

ex. $A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$

$$\Rightarrow \quad AB = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O \quad \text{zero matrix}$$

$$BA = \begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -8 & 4 \end{pmatrix}$$

Also, note from the above that $AB = O$ does not imply that $A = O$ or $B = O$ or both A and B are zero matrices.

Suppose $A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow \quad AB = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 4 & 3 \end{pmatrix}$$

$$AC = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 4 & 3 \end{pmatrix}$$

Therefore, when the products AB and AC are equal, it does not imply that $B = C$.

Linear simultaneous equations are easily represented in matrix form. Consider a system of 4 equations in 3 unknowns, i.e.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 = b_4$$

The 4 X 3 system of equations is equivalent to

$$A\underline{x} = \underline{b}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$ Coefficient matrix

$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ vector of constants

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ vector of unknowns

To demonstrate the equivalence of $A\underline{x} = \underline{b}$ and the 4 equations in 3 unknowns

$$A\underline{x} = \underline{b}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$\underset{4 \times 1}{\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 \end{pmatrix}} = \underset{4 \times 1}{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}}$$

Equality of the above two 4 X 1 matrices (column vectors) produces the original system of 4 equations in 3 unknowns.

ex.
$$\begin{aligned} 2x_1 - x_2 + x_3 &= 5 \\ x_1 + x_2 + 5x_3 &= 0 \\ 6x_2 - 9x_3 &= 7 \end{aligned}$$

is equivalent to $A\underline{x} = \underline{b}$ where

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 5 \\ 0 & 6 & -9 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

In general, any set of m linear equations in n unknowns can be represented in $A\underline{x} = \underline{b}$ form where A is an $m \times n$ matrix, $\underline{x}$ is an $n \times 1$ column vector and $\underline{b}$ is an $m \times 1$ column vector.

The Matrix Inverse

The <u>inverse</u> of an $n \times n$ matrix A, denoted A^{-1}, satisfies

$$A^{-1}A = A^{-1}A = I$$

ex. $$A = \begin{pmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{pmatrix}$$

$$\Rightarrow \quad A^{-1}A = \begin{pmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{pmatrix}\begin{pmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AA^{-1} = \begin{pmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{pmatrix}\begin{pmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix inverse can sometimes be used to solve a system of linear simultaneous equations.

Given $\quad A\underline{x} = \underline{b} \quad$ where $\quad A$ is $n \times n$

$\underline{x}$ is $n \times 1$

$\underline{b}$ is $n \times 1$

Multiplying both sides of $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$ by A^{-1} on the left

$$\Rightarrow \quad A^{-1} A \underset{\sim}{x} = A^{-1} \underset{\sim}{b}$$

$$I \underset{\sim}{x} = A^{-1} \underset{\sim}{b}$$

$$\underset{\sim}{x} = A^{-1} \underset{\sim}{b}$$ is the unique solution to $A\underset{\sim}{x} = \underset{\sim}{b}$.

ex.
$$5x_1 + 8x_2 + x_3 = 2$$
$$2x_2 + x_3 = -1$$
$$4x_1 + 3x_2 - x_3 = 3$$

$A\underset{\sim}{x} = \underset{\sim}{b}$ where $A = \begin{pmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{pmatrix}$$

$$\underset{\sim}{x} = A^{-1} \underset{\sim}{b}$$
$$= \begin{pmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\underset{\sim}{x} = \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} \qquad (x_1 = 3, x_2 = -2, x_3 = 3)$$

The above approach does not work if A^{-1} does not exist. For an $n \times n$ matrix A, if the inverse A^{-1} does exist, then A is a <u>non-singular</u> matrix. otherwise, when A does not possess an inverse it is called a <u>singular</u> matrix.

Recall from the earlier treatment of a square system of linear equations, a unique solution to $A\underset{\sim}{x} = \underset{\sim}{b}$ exists when the determinant of A was non-zero.

Consequently,

Given $A\underline{x} = \underline{b}$, A is $n \times n$

There will be a unique solution provided

$$\left.\begin{array}{l} |A| \neq 0 \\ A^{-1} \text{ exists} \\ A \text{ is non-singular} \end{array}\right\} \text{equivalent statements}$$

Conversely, there will not be a unique solution if

$$\left.\begin{array}{l} |A| = 0 \\ A^{-1} \text{ does not exist} \\ A \text{ is singular} \end{array}\right\} \text{equivalent statements}$$

Note, when A is singular, the system of equations $A\underline{x} = \underline{b}$ can be either consistent (infinite solutions) or inconsistent (no solution).

Clearly, when A is non-singular, the system of equations $A\underline{x} = \underline{b}$ is always consistent because $\underline{x} = A^{-1}\underline{b}$ is the unique solution.

ex. Investigate the nature of the solutions to the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 - x_2 - x_3 &= -3 \\ -4x_1 + 3x_2 + 5x_3 &= 17 \end{aligned}$$

$A\underline{x} = \underline{b}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -4 & 3 & 5 \end{vmatrix} = -6$$

$\Rightarrow A\underline{x} = \underline{b}$ has a unique solution

$$A^{-1} = \begin{pmatrix} 1/3 & 1/3 & 0 \\ 1 & -3/2 & -1/2 \\ -1/3 & 7/6 & 1/2 \end{pmatrix}$$

$$\underline{x} = A^{-1}\underline{b}$$

$$= \begin{pmatrix} 1/3 & 1/3 & 0 \\ 1 & -3/2 & -1/2 \\ -1/3 & 7/6 & 1/2 \end{pmatrix} \begin{pmatrix} 6 \\ -3 \\ 17 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$a + 2b + 3c = 0$

$4a + 5b + 6c = -1$

$7a + 8b + 9c = 2$

$A\underline{x} = \underline{b}$

$|A| = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 0$ $A\underline{x} = \underline{b}$ does not have a unique solution (A^{-1} does not exist. A is singular)

The right hand side vector of constants determines whether the equations are consistent with infinite solutions or inconsistent with no solution.

Finding the Inverse of a Matrix

Def. Given an $n \times n$ matrix A with elements a_{ij}, the cofactor of element a_{ij} is a scalar obtained by multiplying $(-1)^{i+j}$ and the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A.

$$\operatorname{cof}(a_{ij}) = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & & & & & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & & & & \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$$

ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$\operatorname{cof}(a_{21}) = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -(18-24) = 6$$

$$\operatorname{cof}(a_{33}) = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 1(5-8) = -3$$

The <u>cofactor matrix</u> A^c with elements c_{ij} is the matrix of cofactors of the elements of A.

ex, $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $A^c = \begin{pmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{pmatrix}$

The determinant of an nxn matrix A can be evaluated by cofactor expansion.

$$|A| = \sum_{k=1}^{n} a_{kj} c_{kj}, \quad j = 1, 2, ---, n \quad \text{Column Expansion}$$

or $$|A| = \sum_{k=1}^{n} a_{ik} c_{ik}, \quad i = 1, 2, ----, n \quad \text{Row Expansion}$$

ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Column 1 Expansion: $|A| = \sum_{k=1}^{3} a_{k1} c_{k1} = a_{11}c_{11} + a_{21}c_{21} + a_{31}c_{31}$

$= 1(-3) + 4(6) + 7(-3)$

$= 0$

Row 2 Expansion: $|A| = \sum_{k=1}^{3} a_{2k} c_{2k} = a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23}$

$= 4(6) + 5(-12) + 6(6)$

$= 0$

Def. The <u>adjoint</u> of an $n \times n$ matrix A is the transpose of the cofactor matrix.

$$\text{Adj } A = (A^c)^T$$

ex.

$$A = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{pmatrix}, \quad A^c = \begin{pmatrix} 18 & 16 & -11 \\ 0 & 16 & -8 \\ -6 & -16 & 17 \end{pmatrix}$$

$$\text{Adj } A = \begin{pmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{pmatrix}$$

The adjoint matrix satisfies the following condition:

$$A \cdot \text{Adj } A = \text{Adj } A \cdot A = |A|\, I$$

Therefore, if $|A| \neq 0$, i.e. A is non-singular

$$\Rightarrow \quad A \cdot \frac{\text{Adj } A}{|A|} = \frac{\text{Adj } A}{|A|} \cdot A = I$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} \cdot \text{Adj } A \qquad \text{provided } |A| \neq 0$$

ex. $A = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{pmatrix}$, $\quad |A| = 48$

$$A^{-1} = \frac{1}{48} \begin{pmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{pmatrix}$$

check:

$$A^{-1} A = \frac{1}{48} \begin{pmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{pmatrix}$$

$$= \frac{1}{48} \begin{pmatrix} 48 & 0 & 0 \\ 0 & 48 & 0 \\ 0 & 0 & 48 \end{pmatrix} = I$$

General Properties of Determinants

1. The value of a determinant is not altered if its rows are written as columns in the same order.

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 0 \\ 0 & 4 & 2 \end{vmatrix}$$

2. If any two rows (or columns) of a determinant are interchanged, the value of the determinant is multiplied by -1.

$$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

3. A factor of the elements of any row (or column) can be placed before the determinant.

$$\begin{vmatrix} 4 & 6 & 1 \\ 3 & -9 & 2 \\ -1 & 12 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 2\cdot 3 & 1 \\ 3 & -3\cdot 3 & 2 \\ -1 & 4\cdot 3 & 5 \end{vmatrix} = 3 \begin{vmatrix} 4 & 2 & 1 \\ 3 & -3 & 2 \\ -1 & 4 & 5 \end{vmatrix}$$

4. If all the elements of one row (or column) of a determinant are multiplied by the same factor k, the value of the determinant is k times the value of the original determinant.

$$\text{Given} \quad \begin{vmatrix} 2 & 5 & 7 \\ 1 & 4 & 0 \\ 5 & 2 & 3 \end{vmatrix} \quad \Rightarrow \quad \begin{vmatrix} 3\cdot 2 & 3\cdot 5 & 3\cdot 7 \\ 1 & 4 & 0 \\ 5 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & 5 & 7 \\ 1 & 4 & 0 \\ 5 & 2 & 3 \end{vmatrix}$$

5. If corresponding elements of two rows (or columns) of a determinant are proportional, the value of the determinant is zero.

$$\begin{vmatrix} 4 & -2 & 1 \\ 12 & -6 & 3 \\ -1 & 12 & 5 \end{vmatrix} = 3\begin{vmatrix} 4 & -2 & 1 \\ 4 & -2 & 1 \\ -1 & 12 & 5 \end{vmatrix} = 3(-1)\begin{vmatrix} 4 & -2 & 1 \\ 4 & -2 & 1 \\ -1 & 12 & 5 \end{vmatrix} = 0$$

6. The value of a determinant remains unchanged if the elements of one row (or column) are altered by adding to them any constant multiple of the corresponding element in any other row (or column).

$$\begin{vmatrix} -6 & 21 & -30 \\ 1 & -3 & 5 \\ 2 & 7 & -4 \end{vmatrix} = \begin{vmatrix} -6+1\cdot 7 & 21-3\cdot 7 & -30+5\cdot 7 \\ 1 & -3 & 5 \\ 2 & 7 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 5 \\ 1 & -3 & 5 \\ 2 & 7 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 5 \\ 0 & -3 & 0 \\ 2 & 7 & -4 \end{vmatrix}$$

EGN 3420 Exam I Name______________

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SHOW ALL WORK!

Problem 3 (20pts)

For the system of equations $A\underline{x} = \underline{b}$ below

a) Find the inverse of matrix A (if it exists) using the method of cofactors.

b) Use the inverse A^{-1} (if it exists) to find the solution to the system of equations.

$$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 6 \\ 2x & - & 3y & - & z & = & -2 \\ 5x & + & 4y & - & 6z & = & 3 \end{array}$$

a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ 5 & 4 & 6 \end{pmatrix}$

$\text{Cof}(A) = \begin{pmatrix} 22 & 7 & 23 \\ 24 & -21 & 6 \\ 7 & 7 & -7 \end{pmatrix}$

$\text{Adj}\, A = \begin{pmatrix} 22 & 24 & 7 \\ 7 & -21 & 7 \\ 23 & 6 & -7 \end{pmatrix}$

$|A| = 105$

$A^{-1} = \frac{1}{|A|} \text{Adj}\, A$

$= \frac{1}{105} \begin{pmatrix} 22 & 24 & 7 \\ 7 & -21 & 7 \\ 23 & 6 & 7 \end{pmatrix}$

b) $x = A^{-1} \underline{b}$

$= \frac{1}{105} \begin{pmatrix} 22 & 24 & 7 \\ 7 & -21 & 7 \\ 23 & 6 & 7 \end{pmatrix} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$

$= \frac{1}{105} \begin{pmatrix} 105 \\ 105 \\ 105 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Ans. $x = 1$

$y = 1$

$z = 1$

Evaluating Determinants

ex. $A = \begin{pmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{pmatrix}$, Find $|A|$

$$|A| = \begin{array}{c c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} & \\ \begin{vmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{vmatrix} & \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

$$= \begin{array}{c c} \begin{array}{ccc} C_1' & C_2' & C_3' \end{array} & \\ \begin{vmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ 0 & 3 & 3 \end{vmatrix} & \begin{array}{l} R_1' = R_1 \\ R_2' = R_2 \\ R_3' = R_1 + R_3 \end{array} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} C_1'' = C_1 & C_2'' = C_2' & C_3'' = C_3' - C_2' \end{array} \\ \begin{vmatrix} 10 & -6 & -3 \\ 6 & -5 & -2 \\ 0 & 3 & 0 \end{vmatrix} \end{array}$$

$$= 3(-1)^{3+2} \begin{vmatrix} 10 & -3 \\ 6 & -2 \end{vmatrix} = -3(-20+18) = 6$$

ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & -6 & -12 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix} = 0$$

ex. $A = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{pmatrix}$

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 6 \\ -2 & 5 & 4 \\ 3 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 6 \\ 0 & 11 & 16 \\ 0 & -8 & -16 \end{vmatrix} = -1\begin{vmatrix} 11 & 16 \\ -8 & -16 \end{vmatrix} = 48$$

ex. $A = \begin{pmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{pmatrix}$

$$|A| = \begin{vmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{vmatrix} = \begin{vmatrix} 10(1) & 10(-6/10) & 10(-9/10) \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{vmatrix}$$

$$= 10\begin{vmatrix} 1 & -3/5 & -9/10 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{vmatrix} = 10\begin{vmatrix} 1 & -3/5 & -9/10 \\ 0 & -7/5 & -16/10 \\ 0 & 15/5 & 30/10 \end{vmatrix}$$

$$= 10\begin{vmatrix} -7/5 & -8/5 \\ 3 & 3 \end{vmatrix} = 10(3)\begin{vmatrix} -7/5 & -8/5 \\ 1 & 1 \end{vmatrix} = 10(3)(-1/5)\begin{vmatrix} 7 & 8 \\ 1 & 1 \end{vmatrix} = -6(7-8)$$

$$= 6$$

ex. Solve

$$\begin{aligned} 3x + y + 2z &= b_1 \\ -2x + 5y + 4z &= b_2 \\ x + 3y + 6z &= b_3 \end{aligned}$$

$A\underline{x} = \underline{b}$, $A = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\underline{x} = A^{-1}\underline{b} = \frac{1}{48}\begin{pmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underline{x} = \frac{1}{48}\begin{pmatrix} 18b_1 & & -6b_3 \\ 16b_1 & +16b_2 & -16b_3 \\ -11b_1 & -8b_2 & +17b_3 \end{pmatrix}$$

Suppose $b_1 = 1$, $b_2 = 0$, $b_3 = -1$

$$\Rightarrow \quad \underline{x} = \frac{1}{48}\begin{pmatrix} 24 \\ 32 \\ -28 \end{pmatrix}, \qquad \begin{aligned} x &= \frac{24}{48} = \frac{1}{2} \\ y &= \frac{32}{48} = \frac{2}{3} \\ z &= \frac{-28}{48} = -\frac{7}{12} \end{aligned}$$

Elementary Row operations on a Matrix

Given

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 - x_2 - x_3 &= 3 \\ 3x_1 + 2x_2 + 4x_3 &= -1 \end{aligned}$$

Solve by Method of Elimination

Solving for x_1 in EQ 1 and substituting the result in EQs 2 & 3

$$x_1 = -x_2 - x_3$$

$$\Rightarrow \quad 2(-x_2 - x_3) - x_2 - x_3 = 0$$

$$\Rightarrow \quad 3(-x_2 - x_3) + 2x_2 + 4x_3 = -1$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ -3x_2 - 3x_3 &= 3 \\ -x_2 + x_3 &= -1 \end{aligned}$$

Solving for x_2 in EQ 2 and substituting the result in EQ 3

$$x_2 = \frac{3 + 3x_3}{-3}$$

$$\Rightarrow \quad -\left(\frac{3+3x_3}{-3}\right) + x_3 = -1$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ -3x_2 - 3x_3 &= 3 \\ 2x_3 &= -2 \end{aligned}$$

$$x_3 = \frac{-2}{2} = -1$$

$$x_2 = \frac{3+3x_3}{-3} = 0$$

$$x_1 = -x_2 - x_3 = 1$$

Consider a matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

There are 3 types of elementary row operations that can be performed on A

I) Multiplication By A Scalar of a Row in A

II) Interchanging (Swapping) Two Rows

III) Scalar Multiplication of A Row Added To Another Row

Each elementary row operation is equivalent to a matrix multiplication of A by an appropriate elementary matrix.

ex. Elementary Row Operation I

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{pmatrix}$$

$$\Rightarrow \quad E_{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

ex.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}$$

$$\Rightarrow \quad E_{\text{II}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

ex.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11}+\lambda a_{31} & a_{12}+\lambda a_{32} & a_{13}+\lambda a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+\beta a_{21} & a_{32}+\beta a_{22} & a_{33}+\beta a_{23} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+\lambda a_{31} & a_{12}+\lambda a_{32} & a_{13}+\lambda a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+\beta a_{21} & a_{32}+\beta a_{22} & a_{33}+\beta a_{23} \end{pmatrix}$$

$$\Rightarrow \quad E_{\text{III}} = \begin{pmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}$$

Suppose we start with an $n \times n$ system of equations $A\underline{x} = \underline{b}$ and perform a sequence of elementary row operations on both sides until the matrix A has been transformed into the identity matrix I

$\Rightarrow$

$$A\underline{x} = \underline{b}$$
$$E_1 A \underline{x} = E_1 \underline{b}$$
$$E_2 E_1 A \underline{x} = E_2 E_1 \underline{b}$$
$$\vdots \qquad\qquad \vdots$$
$$E_k \cdots E_2 E_1 A \underline{x} = E_k \cdots E_2 E_1 \underline{b}$$

where $E_k \text{-----} E_2 E_1 A = I$

$$\Rightarrow \quad I\underline{x} = E_k \text{-----} E_1 \underline{b}$$

$$\underline{x} = E_k \text{-----} E_1 \underline{b}$$

From the definition of matrix inverse, i.e. $A^{-1}A = A^{-1}A = I$ it follows that

$$A^{-1} = E_k \text{-----} E_2 E_1$$

$$= E_k \text{-----} E_2 E_1 I$$

Thus if we start with matrices A and I and use elementary operations $E_1, E_2, \text{---}, E_k$ to transform A into I, the same elementary row operations $E_1, E_2 \text{---}, E_k$ will transform I into A^{-1}.

ex. $A = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{pmatrix}$ Find A^{-1}

$$\begin{pmatrix} 3 & 1 & 2 & 1 & 0 & 0 \\ -2 & 5 & 4 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 6 & 0 & 0 & 1 \\ -2 & 5 & 4 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 6 & 0 & 0 & 1 \\ 0 & 11 & 16 & 0 & 1 & 2 \\ 0 & -8 & -16 & 1 & 0 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 6 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1/8 & 0 & 3/8 \\ 0 & 11 & 16 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 6 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1/8 & 0 & 3/8 \\ 0 & 0 & -6 & 11/8 & 1 & -17/8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 6 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1/8 & 0 & 3/8 \\ 0 & 0 & 1 & -11/48 & -1/6 & 17/48 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 66/48 & 1 & -54/48 \\ 0 & 1 & 0 & 16/48 & 2/6 & -16/48 \\ 0 & 0 & 1 & -11/48 & -1/6 & 17/48 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 18/48 & 0 & -6/48 \\ 0 & 1 & 0 & 16/48 & 2/6 & -16/48 \\ 0 & 0 & 1 & -11/48 & -1/6 & 17/48 \end{pmatrix}$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} 18/48 & 0 & -6/48 \\ 16/48 & 2/6 & -16/48 \\ -11/48 & -1/6 & 17/48 \end{pmatrix} = \frac{1}{48}\begin{pmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{pmatrix}$$

Elementary row operations can be used to solve a system of equations $A\underline{x} = \underline{b}$.

Def. The <u>augmented</u> <u>matrix</u> $(A \vdots \underline{b})$ is the matrix consisting of A supplemented with one additional column, the $\underline{b}$ vector. Thus if A is $m \times n$, then $(A \vdots \underline{b})$ is $m \times n{+}1$.

If A is $n \times n$, then elementary row operations on $(A \vdots \underline{b})$ resulting in the left side $n \times n$ matrix in upper triangular form is equivalent to a Gauss Elimination.

ex. $A\underline{x} = \underline{b}$ where $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 2 & 4 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$

$$(A \vdots \underline{b}) = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 3 & 2 & 4 & -1 \end{array}\right) \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & -1 & 1 & -1 \end{array}\right) \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right) \end{array}$$

The original system of equations $A\underline{x} = \underline{b}$ has been transformed into a new system $\hat{A}\underline{x} = \hat{\underline{b}}$ where $\hat{A}$ is in upper triangular form. Because $\hat{A}\underline{x} = \hat{\underline{b}}$ was obtained from $A\underline{x} = \underline{b}$ by a sequence of elementary row operations, both systems of equations have the same solutions. From $\hat{A}\underline{x} = \hat{\underline{b}}$

$$\Rightarrow \quad \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ -3x_2 - 3x_3 &= 3 \\ 2x_3 &= -2 \end{aligned}$$

soln. $x_3 = -1$

$x_2 = 0$

$x_1 = 1$

Gauss Jordan method

Gauss Jordan is similar to Gauss Elimination except there is no back substitution required because Gauss Jordan transforms $(A \vdots \underline{b})$ into $(I \vdots \hat{\underline{b}})$ when solving $A\underline{x} = \underline{b}$.

ex. $A\underline{x} = \underline{b}$ where $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 2 & 4 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$

$$(A \vdots \underline{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 3 & 2 & 4 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \end{array}$$

soln. $x_1 = 1$

$x_2 = 0$

$x_3 = -1$

ex. $A\underline{x} = \underline{b}$ where $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 6 \\ -1 \\ 0 \end{pmatrix}$

$$(A \vdots \underline{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 0 & -3 & -1 \\ 1 & -2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -4 & -9 & -13 \\ 0 & -4 & -2 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 1/2 & 3/2 \\ 0 & -4 & -9 & -13 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 1/2 & 3/2 \\ 0 & 0 & -7 & -7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 1/2 & 3/2 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

soln. $x_1 = 1$

$x_2 = 1$

$x_3 = 1$

Gauss Seidel

Jacobi method is an iterative technique for solving $A\underline{x} = \underline{b}$ where $A - n \times n$ and $a_{ii} \neq 0$

ex.
$$3x_1 + x_2 + x_3 = 2$$
$$2x_1 - 4x_2 - x_3 = 3$$
$$x_1 + 2x_2 + 4x_3 = -3$$

Each equation in $A\underline{x} = \underline{b}$ can be solved for x_i

$\Rightarrow$
$$x_1 = \frac{1}{3}(2 - x_2 - x_3)$$
$$x_2 = \frac{1}{4}(-3 + 2x_1 - x_3)$$
$$x_3 = \frac{1}{4}(-3 - x_1 - 2x_2)$$

Initial Guess: $x_1^0 = x_2^0 = x_3^0 = 0$

$$x_1^1 = \frac{1}{3}(2 - x_2^0 - x_3^0) = 2/3$$
$$x_2^1 = \frac{1}{4}(-3 + 2x_1^0 - x_3^0) = -3/4$$
$$x_3^1 = \frac{1}{4}(-3 - x_1^0 - 2x_2^0) = -3/4$$

$$x_1^2 = \frac{1}{3}(2 - x_2^1 - x_3^1) = 7/6$$
$$x_2^2 = \frac{1}{4}(-3 + 2x_1^1 - x_3^1) = -11/48$$
$$x_3^2 = \frac{1}{4}(-3 - x_1^1 - 2x_2^1) = -13/24$$

$$\underline{x} = \begin{pmatrix} 2/3 \\ -3/4 \\ -3/4 \end{pmatrix} + \begin{pmatrix} 0 & -1/3 & -1/3 \\ 1/2 & 0 & -1/4 \\ -1/4 & -1/2 & 0 \end{pmatrix} \underline{x}$$

Initial Guess: $\underline{x}^0 = \underline{0}$

$$\underline{x}^1 = \begin{pmatrix} 2/3 \\ -3/4 \\ -3/4 \end{pmatrix} + \begin{pmatrix} 0 & -1/3 & -1/3 \\ 1/2 & 0 & -1/4 \\ -1/4 & -1/2 & 0 \end{pmatrix} \underline{0}$$
$$= \begin{pmatrix} 2/3 \\ -3/4 \\ -3/4 \end{pmatrix}$$

$$\underline{x}^2 = \begin{pmatrix} 2/3 \\ -3/4 \\ -3/4 \end{pmatrix} + \begin{pmatrix} 0 & -1/3 & -1/3 \\ 1/2 & 0 & -1/4 \\ -1/4 & -1/2 & 0 \end{pmatrix} \underline{x}^1$$
$$= \begin{pmatrix} 2/3 \\ -3/4 \\ -3/4 \end{pmatrix} + \begin{pmatrix} 0 & -1/3 & -1/3 \\ 1/2 & 0 & -1/4 \\ -1/4 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 2/3 \\ -3/4 \\ -3/4 \end{pmatrix}$$
$$= \begin{pmatrix} 7/6 \\ -11/48 \\ -13/24 \end{pmatrix}$$

$$x_1^3 = \frac{1}{3}\left(2 - x_2^2 - x_3^2\right) = 133/144$$

$$x_2^3 = \frac{1}{4}\left(-3 + 2x_1^2 - x_3^2\right) = -3/96$$

$$x_3^3 = \frac{1}{4}\left(-3 - x_1^2 - 2x_2^2\right) = -89/96$$

For the (i+1)st iteration

$$x_1^{i+1} = \frac{1}{3}\left(2 - x_2^i - x_3^i\right)$$

$$x_2^{i+1} = \frac{1}{4}\left(-3 + 2x_1^i - x_3^i\right)$$

$$x_3^{i+1} = \frac{1}{4}\left(-3 - x_1^i - 2x_2^i\right)$$

$i = 0, 1, 2, 3, \ldots$

$$\underline{x}^3 = \begin{pmatrix} 2/3 \\ -3/4 \\ -3/4 \end{pmatrix} + \begin{pmatrix} 0 & -1/3 & -1/3 \\ 1/2 & 0 & -1/4 \\ -1/4 & -1/2 & 0 \end{pmatrix} \underline{x}^2$$

$$= \begin{pmatrix} 2/3 \\ -3/4 \\ -3/4 \end{pmatrix} + \begin{pmatrix} 0 & -1/3 & -1/3 \\ 1/2 & 0 & -1/4 \\ -1/4 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 7/6 \\ -11/48 \\ -13/24 \end{pmatrix}$$

$$= \begin{pmatrix} 133/24 \\ -3/96 \\ -89/96 \end{pmatrix}$$

$$\underline{x}^{i+1} = \begin{pmatrix} 2/3 \\ -3/4 \\ -3/4 \end{pmatrix} + \begin{pmatrix} 0 & -1/3 & -1/3 \\ 1/2 & 0 & -1/4 \\ -1/4 & -1/2 & 0 \end{pmatrix} \underline{x}^i$$

$i = 0, 1, 2, \ldots$

The stop criterion is based on the approximate relative error for each variable x_1, x_2, x_3

$$e_{A,i} = \left| \frac{x_i^{new} - x_i^{old}}{x_i^{new}} \right| \times 100 \qquad i = 1, 2, 3$$

Stop when $\max\left[(e_{A,1}), (e_{A,2}), (e_{A,3})\right] < e_s$

$$\Rightarrow e_{A,1} = \left| \frac{x_1^3 - x_1^2}{x_1^3} \right| \times 100 = \left| \frac{133/144 - 7/6}{133/144} \right| \times 100 = 26.3\%$$

$$e_{A,2} = \left| \frac{x_2^3 - x_2^2}{x_2^3} \right| \times 100 = \left| \frac{-3/96 - -11/48}{-3/96} \right| \times 100 = 633.3\%$$

$$e_{A,3} = \left| \frac{x_3^3 - x_3^2}{x_3^3} \right| \times 100 = \left| \frac{-89/96 - -13/24}{-89/96} \right| \times 100 = 41.6\%$$

The Gauss-Seidel Method is similar to the Jacobi Method except that the updated values for x_1, x_2, x_3 are used immediately instead of waiting for the next iteration.

In the previous example,

$$x_1^{i+1} = \frac{1}{3}\left(2 - x_2^{i} - x_3^{i}\right)$$

$$x_2^{i+1} = \frac{1}{4}\left(-3 + 2x_1^{i+1} - x_3^{i}\right)$$

$$x_3^{i+1} = \frac{1}{4}\left(-3 - x_1^{i+1} - 2x_2^{i+1}\right)$$

$i = 0, 1, 2, 3, \ldots$

$i=0$:

$$x_1^{1} = \frac{1}{3}\left(2 - x_2^{0} - x_3^{0}\right) = \frac{1}{3}\left(2 - 0 - 0\right) = 2/3$$

$$x_2^{1} = \frac{1}{4}\left(-3 + 2x_1^{1} - x_3^{0}\right) = \frac{1}{4}\left[-3 + 2(2/3) - 0\right] = -5/12$$

$$x_3^{1} = \frac{1}{4}\left(-3 - x_1^{1} - 2x_2^{1}\right) = \frac{1}{4}\left[-3 - 2/3 - 2(-5/12)\right] = -17/24$$

$i=1$:

$$x_1^{2} = \frac{1}{3}\left(2 - x_2^{1} - x_3^{1}\right) = \frac{1}{3}\left(2 - \frac{-5}{12} - \frac{-17}{24}\right) = 75/72$$

$$x_2^{2} = \frac{1}{4}\left(-3 + 2x_1^{2} - x_3^{1}\right) = \frac{1}{4}\left[-3 + 2(75/72) - {-17/24}\right] = -15/288$$

$$x_3^{2} = \frac{1}{4}\left(-3 - x_1^{2} - 2x_2^{2}\right) = \frac{1}{4}\left[-3 - \frac{75}{72} - 2(-15/288)\right] = -577/576$$

The Gauss Seidel Method is usually faster than the Jacobi Method when they converge to the solution.

Convergence Criterion for Gauss Seidel Method

A sufficient but not necessary condition for convergence of the Gauss Seidel Method is:

The absolute value of the diagonal coefficient in each of the equations be larger than the sum of the absolute values of the other coefficients in the equation.

That is, Given $A\underline{x} = \underline{b}$, Gauss-Seidel will converge if

$$|a_{ii}| > \sum_{\substack{j=1 \\ (j \neq i)}}^{n} |a_{ij}|$$

The system of equations are called <u>diagonally dominant</u> when the above condition holds.

example

$$\begin{aligned} x_1 \qquad + x_3 &= 4 \\ x_1 + x_2 - x_3 &= 0 \\ 3x_1 \qquad - 2x_3 &= -3 \end{aligned}$$

This system $A\underline{x} = \underline{b}$ is not diagonally dominant.

The Gauss Seidel Method fails to converge.

$A\underline{x} = \underline{b}$ can be transformed into a diagonally dominant system as follows.

$$\begin{aligned} 3x_1 \quad\quad\quad - 2x_3 &= -3 & \quad R_3 \\ x_1 + 4x_2 - 2x_3 &= 3 & \quad 4R_2 - R_3 \\ -x_2 + 2x_3 &= 4 & \quad R_1 - R_2 \end{aligned}$$

The algorithm for the above system is

$$x_1^{i+1} = -1 + \frac{2}{3} x_3^{i}$$

$$x_2^{i+1} = \frac{3}{4} - \frac{1}{4} x_1^{i+1} + \frac{1}{2} x_3^{i}$$

$$x_3^{i+1} = 2 + \frac{1}{2} x_2^{i+1}$$

For $x_1^{0} = x_2^{0} = x_3^{0} = 0$

$$x_1^{1} = -1 + \frac{2}{3} x_3^{0} = -1 + \frac{2}{3}(0) = -1$$

$$x_2^{1} = \frac{3}{4} - \frac{1}{4} x_1^{1} + \frac{1}{2} x_3^{0} = \frac{3}{4} - \frac{1}{4}(-1) + \frac{1}{2}(0) = 1$$

$$x_3^{1} = 2 + \frac{1}{2} x_2^{1} = 2 + \frac{1}{2}(1) = \frac{5}{2}$$

$$x_1^{2} = -1 + \frac{2}{3} x_3^{1} = -1 + \frac{2}{3}\left(\frac{5}{2}\right) = \frac{2}{3}$$

$$x_2^{2} = \frac{3}{4} - \frac{1}{4} x_1^{2} + \frac{1}{2} x_3^{1} = \frac{3}{4} - \frac{1}{4}\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{5}{2}\right) = \frac{11}{6}$$

$$x_3^{2} = 2 + \frac{1}{2} x_2^{2} = 2 + \frac{1}{2}\left(\frac{11}{6}\right) = \frac{35}{12}$$

etc

THE ECHELON FORM

The echelon form of the augmented matrix $(\mathbf{A}|\underline{\mathbf{b}})$ is obtained by performing a sequence of elementary row operations. Ones are placed in the i^{th} row and i^{th} column element and zeros below. The echelon form results when the procedure terminates. The echelon form reveals the nature of the solutions to the original system of equations $A\underline{x} = \underline{b}$ where A is the m by n coefficient matrix. Several examples follow.

Example 1 A system of 4 equations in 3 unknowns (m=4, n=3)

$$\begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ 2x & - & y & - & z & = & 3 \\ 3x & - & 2y & + & z & = & 2 \\ x & + & 4y & + & 2z & = & -1 \end{array}$$

$$(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 3 & -2 & 1 & 2 \\ 1 & 4 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & -5 & -2 & 2 \\ 0 & 3 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -5 & -2 & 2 \\ 0 & 3 & 1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The final matrix is the echelon form of the augmented matrix. The original system has reduced to a consistent system of 3 equations with 3 unknowns. Hence, one of the 4 original equations was redundant (linearly dependent) and the resulting 3 by 3 system has a unique solution. The solution can be obtained by backward substitution in the 3 linearly independent equations obtained from the echelon form as in a Gauss Elimination. Alternatively, the Gauss-Jordan method can be used to insert zeros above the ones resulting in the identity matrix in the first 3 columns and the solution in the 4^{th} column. Both methods are illustrated below.

I. Gauss Elimination

$$\begin{array}{rclcll} & z = & -1 & & & \text{from the 3rd equation} \\ y+z & = & -1 & \Rightarrow & y=0 & \text{from the 2nd equation} \\ x+y+z & = & 0 & \Rightarrow & x=1 & \text{from the 1st equation} \end{array}$$

II. Gauss-Jordan

$$(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow \quad x = 1, \quad y = 0, \quad z = -1$$

There is a convenient matrix function in MATLAB which converts a matrix to its row reduced echelon form. This is similar to the echelon form discussed here except that zeros appear above as well as below the ones in the respective columns. It is actually more convenient to have the echelon form expressed in this way because the solution(s) to the system of equations become more apparent.

In Example 1 the echelon form revealed a consistent system with a unique solution. Additional work was required to find the solution. If we utilize the MATLAB function "rref" to obtain the row reduced echelon form of ($\mathbf{A}|\underline{\mathbf{b}}$), the solution is obtained at the same time.

```
A =

     1     1     1     0
     2    -1    -1     3
     3    -2     1     2
     1     4     2    -1

EDU» rref(A)

ans =

     1     0     0     1
     0     1     0     0
     0     0     1    -1
     0     0     0     0
```

The nonzero rows of the row reduced echelon form are identical to those in the matrix obtained by the Gauss-Jordan method.

Example 2 A system of 5 equations in 3 unknowns (m=5, n=3)

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & - & x_2 & - & x_3 & = & 3 \\ & & 2x_2 & - & x_3 & = & 4 \\ 3x_1 & - & 2x_2 & + & x_3 & = & -1 \\ 4x_1 & - & 3x_2 & + & x_3 & = & -1 \end{array}$$

The echelon form of the augmented matrix is found the same way as in Example 1.

$$(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{rrr:r} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 0 & 2 & -1 & 4 \\ 3 & -2 & 1 & -1 \\ 4 & -3 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{rrr:r} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -5 & -2 & -1 \\ 0 & -7 & -3 & -1 \end{array}\right) \sim \left(\begin{array}{rrr:r} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 4 & -8 \end{array}\right) \sim \left(\begin{array}{rrr:r} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The echelon form of the augmented matrix indicates that 2 of the original 5 equations were redundant and the remaining 3 equations comprise a consistent system with a unique solution.

Example 3 A system of 6 equations in 3 unknowns (m=6, n=3)

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & - & x_2 & - & x_3 & = & 3 \\ & & 2x_2 & - & x_3 & = & 4 \\ 3x_1 & - & 2x_2 & + & x_3 & = & -1 \\ 4x_1 & - & 3x_2 & + & x_3 & = & -1 \\ 10x_1 & - & 3x_2 & + & x_3 & = & 4 \end{array}$$

The augmented matrix and its echelon form are given below.

$$(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{rrr:r} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 0 & 2 & -1 & 4 \\ 3 & -2 & 1 & -1 \\ 4 & -3 & 1 & -1 \\ 10 & -3 & 1 & 4 \end{array}\right) \sim\ldots\ldots\sim \left(\begin{array}{rrr:r} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

In this case the echelon form implies the original system of equations was inconsistent as a result of the last line, which in equation form reads $0x + 0y + 0z = -1$. In fact, the system of equations in Example 3 are the same as in Example 2 with one additional equation. The left side of the added equation is simply the sum of the left side of all the equations above it. For the new system of equations to be consistent, it follows that the right hand side constant in the last equation must also be the sum of the constants in the original 5 equations. However; it is not and therefore the complete original system of equations is inconsistent.

Example 4 A system of m = 6 equations in n = 5 unknowns

$$\begin{array}{ccccccccccccc}
a & + & b & + & c & - & d & - & 3e & = & 3 \\
a & - & b & - & c & - & 3d & + & e & = & -1 \\
 & & 2b & + & c & + & d & - & 2e & = & 4 \\
a & & & + & 2c & & & - & 5e & = & 1 \\
2a & - & b & + & c & - & 3d & - & 4e & = & 0 \\
3a & - & 2b & & & - & 6d & - & 3e & = & -1
\end{array}$$

$$(\mathbf{A}|\underline{\mathbf{b}}) = \begin{pmatrix} 1 & 1 & 1 & -1 & -3 & 3 \\ 1 & -1 & -1 & -3 & 1 & -1 \\ 0 & 2 & 1 & 1 & -2 & 4 \\ 1 & 0 & 2 & 0 & -5 & 1 \\ 2 & -1 & 1 & -3 & -4 & 0 \\ 3 & -2 & 0 & -6 & -3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & -1 & -3 & 3 \\ 0 & -2 & -2 & -2 & 4 & -4 \\ 0 & 2 & 1 & 1 & -2 & 4 \\ 0 & -1 & 1 & 1 & -2 & -2 \\ 0 & -3 & -1 & -1 & 2 & -6 \\ 0 & -5 & -3 & -3 & 6 & -10 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & -1 & -3 & 3 \\ 0 & 1 & 1 & 1 & -2 & 2 \\ 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 2 & 2 & -4 & 0 \\ 0 & 0 & 2 & 2 & -4 & 0 \\ 0 & 0 & 2 & 2 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & -1 & -3 & 3 \\ 0 & 1 & 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this example, the echelon form has reduced to an equivalent system with 3 equations and 5 unknowns. Since the last three rows consist entirely of zeros, the original system of equations is consistent and equivalent to a system of 3 equations in 5 unknowns. In this case, two of the five variables may be chosen arbitrarily. The criterion for selecting which variables may be arbitrary is discussed later. Regardless of that decision, an infinite number of solutions are possible due to the existence of arbitrary unknowns.

One additional example is included to illustrate the case where there are fewer equations than unknowns.

Example 5 A system of 4 equations in 5 unknowns (m=4, n=5)

$$
\begin{array}{rcrcrcrcrcr}
x_1 & + & x_2 & + & x_3 & - & x_4 & + & 3x_5 & = & 3 \\
2x_1 & - & x_2 & + & x_3 & - & 2x_4 & - & 2x_5 & = & 2 \\
 & & x_2 & - & x_3 & + & 2x_4 & + & 2x_5 & = & 0 \\
-x_1 & + & 3x_2 & - & x_3 & + & 3x_4 & + & 7x_5 & = & 1
\end{array}
$$

The row reduced echelon form from MATLAB is given below.

$$
\text{rref}(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0.5 & 2.5 & 1 \\
0 & 0 & 1 & -1.5 & -0.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Once again the equations are consistent; however, one of the original equations was redundant. Similar to Example 4, a pair of arbitrary variables can be selected from the set $(x_1, x_2, x_3, x_4, x_5)$. As a result, an infinite number of solutions exist.

These previous examples illustrate the rationale for transforming the augmented matrix representing the system of equations into an echelon form. The echelon form is also useful in other ways. Example 6 demonstrates another useful application of the echelon form.

Example 6 Find the value(s) of K for which the system of equations are consistent.

$$
\begin{array}{ccccccccc}
x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \\
2x_1 & - & x_2 & + & x_3 & & & = & 3 \\
 & & 3x_2 & - & 2x_3 & + & 3x_4 & = & 1 \\
3x_1 & - & 3x_2 & + & 4x_3 & - & 2x_4 & = & K
\end{array}
$$

Before we look at the echelon form of the augmented matrix, its a good idea to check the coefficient matrix. For the 4 by 4 system denoted by $\mathbf{A}\underline{\mathbf{x}} = \underline{\mathbf{b}}$, if $|\mathbf{A}|$ were other than zero, the matrix inverse $\mathbf{A}^{-1}$ would exist as would a unique solution for any value of K. Consequently, the equations would be consistent for all values of K. After confirming that $|\mathbf{A}| = 0$, we turn to the echelon form for the answer.

$$
(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & 0 & 3 \\ 0 & 3 & -2 & 3 & 1 \\ 3 & -3 & 4 & -2 & K \end{array}\right) \sim \left(\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & -2 & -1 \\ 0 & 3 & -2 & 3 & 1 \\ 0 & -6 & 1 & -5 & K-6 \end{array}\right)
$$

$$
\sim \left(\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 3 & -1 & K-4 \end{array}\right) \sim \left(\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & K-4 \end{array}\right)
$$

A consistent system of equations with infinite solutions results when the last row of the echelon form is all zeros. Therefore, the only solution is K = 4.

ARBITRARY UNKNOWNS

The echelon form of the augmented matrix confirms the existence of arbitrary unknowns, i.e. a consistent system of equations in which one or more variables can be chosen arbitrarily. There are several ways to establish if indeed a certain variable can be included in the subset of arbitrary unknowns. A few simple examples illustrate the point.

Example 1 For the system of equations below, establish the existence of 1 arbitrary unknown and determine if x, y, and z can each be arbitrary.

$$\begin{aligned} x + y + z &= 0 \\ 2x - y - z &= 3 \\ x + 4y + 4z &= -3 \\ x - 2y - 2z &= 3 \end{aligned}$$

$$(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 1 & 4 & 4 & -3 \\ 1 & -2 & -2 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & 3 & 3 & -3 \\ 0 & -3 & -3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The 4 by 4 system has been reduced to a 2 by 3 system. That is, there are only 2 linearly independent equations in the 3 variables (unknowns) and hence one of the variables is arbitrary. These equations are

$$\begin{aligned} x + y + z &= 0 \\ y + z &= -1 \end{aligned}$$

Now, suppose we try to make z the arbitrary unknown. This requires that solutions for x and y be expressed in terms of z. This can be attempted in one of two ways. One approach is to simply transfer all terms involving z over to the right side of the equations and consider z as a parameter. This yields

$$\begin{aligned} x + y &= -z \\ y &= -1 - z \end{aligned}$$

All that remains is substituting $y = -1 - z$ into the first equation and then solving for x. The final solution can be expressed as

$$x = 1, \quad y = -1-z, \quad z = \text{arbitrary}$$

Clearly, an infinite number of solutions exist since there is a different solution for each arbitrarily assigned value for z. A slightly different approach involves reformulating the reduced equations from the echelon form as

$$\hat{\mathbf{A}}\hat{\underline{\mathbf{x}}} = \hat{\underline{\mathbf{b}}} \quad \text{where} \quad \hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \hat{\underline{\mathbf{b}}} = \begin{pmatrix} -z \\ -1-z \end{pmatrix}, \quad \hat{\underline{\mathbf{x}}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. as a new system in matrix form with modified coefficient matrix $\hat{\mathbf{A}}$, constant vector $\hat{\underline{\mathbf{b}}}$, and vector of unknowns $\hat{\underline{\mathbf{x}}}$. The identical solution as given above is easily obtained from the modified augmented matrix

$$\left(\hat{\mathbf{A}}|\hat{\underline{\mathbf{b}}}\right) = \left(\begin{array}{cc|c} 1 & 1 & -z \\ 0 & 1 & -1-z \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1-z \end{array}\right)$$

$$x = 1, \quad y = -1-z, \quad z = \text{arbitrary}$$

The second approach is somewhat more instructive because the modified coefficient matrix $\hat{\mathbf{A}}$ determines whether the variables placed on the right hand side (z in this example) are arbitrary. When $\hat{\mathbf{A}}$, which will always be a square matrix, is nonsingular, $\hat{\mathbf{A}}\hat{\underline{\mathbf{x}}} = \hat{\underline{\mathbf{b}}}$ has a unique solution and the variables moved to the right hand side are indeed arbitrary.

Consider what happens when x is selected to be the arbitrary variable. The reduced 2 by 3 system with x as the arbitrary unknown becomes

$$\hat{\mathbf{A}}\hat{\underline{\mathbf{x}}} = \hat{\underline{\mathbf{b}}} \quad \text{where} \quad \hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \hat{\underline{\mathbf{b}}} = \begin{pmatrix} -x \\ -1 \end{pmatrix}, \quad \hat{\underline{\mathbf{x}}} = \begin{pmatrix} y \\ z \end{pmatrix}$$

and attempting to solve for a solution by Gauss-Jordan gives

$$\left(\hat{\mathbf{A}}|\hat{\underline{\mathbf{b}}}\right) = \left(\begin{array}{cc|c} 1 & 1 & -x \\ 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & x-1 \end{array}\right)$$

In this case, a unique solution for y and z in terms of x was not possible. This comes as no surprise because $\hat{\mathbf{A}}$ is clearly singular. This confirms what was already known from the previous case where z was arbitrary but x was constrained to be 1. In fact, the Gauss-Jordan solution above implies that x-1 = 0 for the equations to be consistent.

For larger systems with arbitrary unknowns, it is essential to check the modified coefficient matrix $\hat{\mathbf{A}}$ before continuing on with a Gauss-Jordan or backward substitution under the assumption that one or more specific variables can be arbitrary. The following system demonstrates this point.

$$\begin{array}{rcrcrcrcrcrcl}
x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & - & 2x_6 & = & 3 \\
 & & x_2 & & & + & x_4 & & & - & 3x_6 & = & 0 \\
x_1 & & & + & x_3 & - & x_4 & + & 2x_5 & + & 4x_6 & = & 4 \\
2x_1 & - & x_2 & - & x_3 & & & & & - & 4x_6 & = & 1 \\
-x_1 & + & 3x_2 & + & 2x_3 & + & x_4 & & & + & 2x_6 & = & 1 \\
 & & x_2 & + & x_3 & - & 2x_4 & - & x_5 & + & 8x_6 & = & 0
\end{array}$$

The echelon form of the augmented matrix is given below.

$$(\mathbf{A}|\underline{\mathbf{b}}) \sim \left(\begin{array}{cccccc|c}
1 & 1 & 1 & 1 & 1 & -2 & 3 \\
0 & 1 & 0 & 1 & 0 & -3 & 0 \\
0 & 0 & 1 & -3 & -1 & 11 & 0 \\
0 & 0 & 0 & 1 & -1 & -3 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

The original 6 by 6 system of equations has been converted to a new 5 by 6 system with a single arbitrary unknown. From the echelon form these equations are

$$
\begin{array}{ccccccccccccc}
x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & - & 2x_6 & = & 3 \\
 & & x_2 & & & + & x_4 & & & - & 3x_6 & = & 0 \\
 & & & & x_3 & - & 3x_4 & - & x_5 & + & 11x_6 & = & 0 \\
 & & & & & & x_4 & - & x_5 & - & 3x_6 & = & -1 \\
 & & & & & & & & x_5 & & & = & 1
\end{array}
$$

Clearly x_5 is not arbitrary. The modified matrix $\hat{\mathbf{A}}$ obtained from the columns of the echelon form with the x_5 column omitted is shown below. Quite obviously its singular, as expected, confirming that x_5 is not arbitrary.

$$
\hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 & 11 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

However, there may be other non-arbitrary variables in addition to x_5 which are not as obvious. For example, suppose we proceed to solve the reduced 5 by 6 system using Gauss-Jordan with x_2 selected as the arbitrary unknown. Observe what happens.

$$
\left(\hat{\mathbf{A}}|\hat{\underline{\mathbf{b}}}\right) = \left(\begin{array}{ccccc:c} 1 & 1 & 1 & 1 & -2 & 3-x_2 \\ 0 & 0 & 1 & 0 & -3 & -x_2 \\ 0 & 1 & -3 & -1 & 11 & 0 \\ 0 & 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc:c} 1 & 1 & 1 & 1 & -2 & 3-x_2 \\ 0 & 1 & -3 & -1 & 11 & 0 \\ 0 & 0 & 1 & 0 & -3 & -x_2 \\ 0 & 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)
$$

$$
\sim \left(\begin{array}{ccccc:c} 1 & 1 & 1 & 1 & -2 & 3-x_2 \\ 0 & 1 & -3 & -1 & 11 & 0 \\ 0 & 0 & 1 & 0 & -3 & -x_2 \\ 0 & 0 & 0 & -1 & 0 & -1+x_2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc:c} 1 & 1 & 1 & 1 & -2 & 3-x_2 \\ 0 & 1 & -3 & -1 & 11 & 0 \\ 0 & 0 & 1 & 0 & -3 & -x_2 \\ 0 & 0 & 0 & 1 & 0 & 1-x_2 \\ 0 & 0 & 0 & 0 & 0 & x_2 \end{array}\right)
$$

The bottom row of zeros in the first 5 columns signifies that a solution for x_6 is impossible when x_2 is chosen to be arbitrary and the Gauss-Jordan method terminates without a solution. Furthermore, for consistency the last row implies that x_2 must be zero, further evidence it can't be arbitrary. The 4th row represents the equation

$$x_5 = 1 - x_2$$

which is consistent with the last row of the echelon form which states that $x_5 = 1$.

In problems of this type, the prudent thing to do is verify that the modified coefficient matrix $\hat{\mathbf{A}}$ is nonsingular before proceeding to find a solution. In the previous example, when x_2 was assumed to be arbitrary, $\hat{\mathbf{A}}$ became

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & -3 & -1 & 11 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

From MATLAB, its easy to verify that $\hat{\mathbf{A}}$ is singular and therefore x_2 should not be chosen as arbitrary.

```
A =

    1    1    1    1   -2
    0    0    1    0   -3
    0    1   -3   -1   11
    0    0    1   -1   -3
    0    0    0    1    0

EDU» det(A)

ans =

    0
```

The same approach applies when more than one variable is arbitrary. To illustrate, consider the system of equations

$$
\begin{array}{cccccccccccc}
x_1 & + & x_2 & + & x_3 & - & x_4 & - & 3x_5 & = & 3 \\
x_1 & - & x_2 & - & x_3 & - & 3x_4 & + & x_5 & = & -1 \\
 & & 2x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & 4 \\
x_1 & & & + & 2x_3 & & & - & 5x_5 & = & 1 \\
2x_1 & - & x_2 & + & x_3 & - & 3x_4 & - & 4x_5 & = & 0 \\
3x_1 & - & 2x_2 & & & - & 6x_4 & - & 3x_5 & = & -1
\end{array}
$$

The echelon form of the augmented matrix has three rows consisting entirely of zeros. The first three non-zero rows are

$$
\left(\begin{array}{ccccc|c}
1 & 1 & 1 & -1 & -3 & 3 \\
0 & 1 & 1 & 1 & -2 & 2 \\
0 & 0 & 1 & 1 & -2 & 0
\end{array}\right)
$$

Alternatively, the row reduced echelon form from MATLAB is shown below.

```
Ab =

     1     1     1    -1    -3     3
     1    -1    -1    -3     1    -1
     0     2     1     1    -2     4
     1     0     2     0    -5     1
     2    -1     1    -3    -4     0
     3    -2     0    -6    -3    -1

EDU» rref(Ab)

ans =

     1     0     0    -2    -1     1
     0     1     0     0     0     2
     0     0     1     1    -2     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
```

In the 3 by 5 system of equations that correspond to either echelon form, there must be 2 arbitrary unknowns. To check if say x_4 and x_5 can be arbitrary, we look at the modified coefficient matrix $\hat{\mathbf{A}}$ that results when the columns for x_4 and x_5 are removed from the first echelon form.

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad |\hat{\mathbf{A}}| = 1$$

Since $\hat{\mathbf{A}}$ is a nonsingular matrix, there is a unique solution to $\hat{\mathbf{A}}\hat{\underline{\mathbf{x}}} = \hat{\underline{\mathbf{b}}}$

$$\text{where} \quad \hat{\underline{\mathbf{x}}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \hat{\underline{\mathbf{b}}} = \begin{pmatrix} 3 + x_4 + 3x_5 \\ 2 - x_4 + 2x_5 \\ -x_4 + 2x_5 \end{pmatrix}$$

The solution is

$$\hat{\underline{\mathbf{x}}} = \left(\hat{\mathbf{A}}\right)^{-1} \hat{\underline{\mathbf{b}}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 + x_4 + 3x_5 \\ 2 - x_4 + 2x_5 \\ -x_4 + 2x_5 \end{pmatrix} = \begin{pmatrix} 1 + 2x_4 + x_5 \\ 2 \\ -x_4 + 2x_5 \end{pmatrix}$$

i.e. $x_1 = 1 + 2x_4 + x_5$, $x_2 = 2$, $x_3 = -x_4 + 2x_5$, $x_4 =$ arbitrary, $x_5 =$ arbitrary

Since we know from the previous solution that x_2 is not arbitrary, it should come as no surprise that any 3 by 3 submatrix formed from the first five columns of either echelon matrix (minus the zero rows) is destined to be singular if it excludes the second column, the one corresponding to x_2. The resulting 3 by 3 matrices obtained from the first echelon form are given below and the reader should verify that they are all singular.

$\begin{pmatrix} -1 & -1 & -3 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{pmatrix}$ x_1 and x_2 columns removed $\qquad$ $\begin{pmatrix} 1 & -1 & -3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix}$ x_2 and x_3 columns removed

$\begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix}$ x_2 and x_4 colums removed $\qquad$ $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ x_2 and x_5 columns removed

The row reduced echelon form is even more explicit as to why x_2 cannot be arbitrary. It is clear from this echelon form that $x_2 = 2$ and hence not arbitrary.

Furthermore, removing the x_2 column from the row reduced echelon form leaves the following matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & -2 & 0 \end{array}\right)$$

Regardless of which additional column is removed for the 2nd arbitrary variable, the resulting 3 by 3 modified coefficient matrix $\hat{\mathbf{A}}$ will be singular, confirming that x_2 cannot be one of the two arbitrary unknowns.

In summary, either echelon form of the original augmented matrix will reveal the existence of arbitrary unknowns. The original m by n system of m equations in n unknowns will be reduced to an m_1 by n system where $m_1 \leq m$ indicating that $m - m_1$ equations from the original system were redundant. If m_1 is less than n, the existence of $n - m_1$ arbitrary unknowns is assured. A particular subset of $n - m_1$ unknowns is arbitrary provided the m_1 by m_1 submatrix of the echelon matrix obtained by removing the columns corresponding to the $n - m_1$ unknowns is nonsingular.

The row reduced echelon form in MATLAB will identify the arbitrary variables directly, i.e. any row with a single 1 in the first n columns implies the variable associated with that column is not arbitrary. For example, in the following 4 by 6 row-reduced echelon form, x_1 and x_5 cannot be arbitrary unknowns.

$$\text{rref}(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 & 3 & 4 & 1 \\ 0 & 0 & 1 & 2 & 7 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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SHOW ALL WORK!

Problem 4 (25pts)

Given the following system of equations, $A\underline{x} = \underline{b}$

$$\begin{aligned} x_1 + x_2 + x_3 + Kx_4 &= 0 \\ -x_1 + x_3 - x_4 &= 0 \\ 2x_1 - x_2 + 3x_3 - 8x_4 &= 0 \\ x_2 + x_3 - 2x_4 &= 0 \end{aligned}$$

Find the value(s) of K for which the above system has a unique solution?

$$|A| = \begin{vmatrix} 1 & 1 & 1 & K \\ -1 & 0 & 1 & -1 \\ 2 & -1 & 3 & -8 \\ 0 & 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & K \\ 0 & 1 & 2 & K-1 \\ 0 & -3 & 1 & -2K-8 \\ 0 & 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & K \\ 0 & 1 & 2 & K-1 \\ 0 & 0 & 7 & K-11 \\ 0 & 0 & -1 & -K-1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & K-1 \\ 0 & 7 & K-11 \\ 0 & -1 & -K-1 \end{vmatrix} = \begin{vmatrix} 7 & K-11 \\ -1 & -K-1 \end{vmatrix} = 7(-K-1) - (-1)(K-11)$$

$$= -6K - 18$$

Setting $|A| = 0$

$\Rightarrow \quad 0 = -6K - 18$

$K = -3$

Ans. All values of K except $K=-3$ result in unique solution