Problem 1 (35 pts)

A) Find \( f_2(x) \), the second order truncated Taylor Series Expansion of the function \( f(x) = x \sin x \) about the point \( x_0. \) Leave your answer in terms of \( x \) and \( x_0. \)

B) For \( x_0 = \pi/2 \) find the true relative error at \( x = 1.1x_0. \)
\[ f(x) = x \sin x \]

A) \[ f_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 \]

\[ f'(x) = x \cos x + \sin x \]

\[ f''(x) = -x \sin x + \cos x + \cos x = 2 \cos x - x \sin x \]

\[ \Rightarrow f_2(x) = x_0 \sin x_0 + (x_0 \cos x_0 + \sin x_0)(x-x_0) + \frac{1}{2}(2 \cos x_0 - x_0 \sin x_0)(x-x_0)^2 \]

B) \[ x_0 = \frac{\pi}{2}, \quad \frac{1}{2} \Delta x = 1.1 \times x_0 \quad \Rightarrow \quad x - x_0 = 0.1 \times x_0 = 0.1 \left( \frac{\pi}{2} \right) \]

\[ f_2(1.1 \times x_0) = \frac{\pi}{2} \sin \frac{\pi}{2} + \left( \frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right)(0.1 \times \frac{\pi}{2}) + \frac{1}{2} \left(2 \cos \frac{\pi}{2} - \frac{\pi}{2} \sin \frac{\pi}{2} \right)(0.01 \frac{\pi^2}{4}) \]

\[ \begin{align*}
&= \frac{\pi}{2} (1) + \left( 0 + 1 \right) \left( 0.1 \frac{\pi}{2} \right) + \frac{1}{2} \left( 0 - \frac{\pi}{2} \right) \left( 0.01 \frac{\pi^2}{4} \right) \\
&= \frac{\pi}{2} + 0.05 \frac{\pi}{2} - \frac{\pi}{4} \left( 0.01 \frac{\pi^2}{4} \right) \\
&= \frac{1}{2} \pi - \frac{\pi}{4} \left( 0.01 \pi^2 \right) \\
&= 0.55 \pi - \frac{\pi^3}{1600} \\
\end{align*} \]

\[ f_2(1.1 \times \frac{\pi}{2}) = 1.7085 \]

\[ e = \frac{f(1.1 \times \frac{\pi}{2}) - f_2(1.1 \times \frac{\pi}{2})}{f(1.1 \times \frac{\pi}{2})} = \frac{\left( \frac{\pi}{2} \sin \frac{\pi}{4} \right) - 1.7085}{f(1.1 \times \frac{\pi}{2})} \]

\[ \approx 1.7066 - 1.7085 \approx -0.0019 \]
Problem 2 (35 pts)

A cantilevered beam extending from its clamped end (x=0) to its free end (x=L) has a maximum deflection $\delta_{\text{max}}$ at the end of the beam, i.e. at x=L. The deflection $\delta$ at location $x=\alpha L$ ($0<\alpha<1$) is related to $\delta_{\text{max}}$ by the equation

$$f(\alpha) = \alpha^4 - 4\alpha^3 + 6\alpha^2 - 3\delta / \delta_{\text{max}} = 0$$

Use the Bisection Method to solve for the value of $\alpha$ at which $\delta / \delta_{\text{max}}$ is equal to 0.75. Start with an initial bracket of $\alpha_L = 0.5$ and $\alpha_U = 1.0$. Complete 3 iterations and fill in the table below. Round all answers to 5 places after the decimal point.

| i | $\alpha_L$ | $\alpha_U$ | $\alpha_R$ | $|e_\alpha|$ |
|---|---|---|---|---|
| 0 | 0.50000 | 1.00000 | 0.75 | ----- |
| 1 | 0.75 | 1.0 | 0.875 | 0.14286 |
| 2 | 0.75 | 0.875 | 0.8125 | 0.07692 |
| 3 | 0.75 | 0.8125 | 0.78125 | 0.04 |
\[ f(x) = x^4 - 4x^3 + 6x^2 - 3x + \frac{5}{2} = 0 \]

\[ d_L = 0.5, \quad d_u = 1 \]

\[ d_R = \frac{d_L + d_u}{2} = 0.75 \]

\[ f(d_L) = f(0.5) = -1.1875 \]

\[ f(d_R) = f(0.75) = -0.2461 \]

\[ f(d_L)f(d_R) = (-)(-) > 0 \]

\[ \Rightarrow d_L = 0.75 \]

\[ d_R = \frac{d_L + d_u}{2} = 0.75 + 1 = 0.875 \]

\[ \left| e_R \right| = \left| \frac{d_R^{new} - d_R^{old}}{d_R^{new}} \right| = \left| \frac{0.875 - 0.75}{0.875} \right| = 0.14286 \]

\[ f(d_L) = f(0.75) = -0.2461 \]

\[ f(d_R) = f(0.875) = 0.2502 \]

\[ f(d_L)f(d_R) = (-)(+) < 0 \]

\[ \Rightarrow d_u = 0.875 \]

\[ d_R = \frac{d_L + d_u}{2} = 0.75 + 0.875 = 0.8125 \]

\[ \left| e_R \right| = \left| \frac{0.8125 - 0.875}{0.8125} \right| = 0.07692 \]

\[ f(d_L) = f(0.75) = -0.2461 \]

\[ f(d_R) = f(0.8125) = 0.0012 \]

\[ f(d_L)f(d_R) = (-)(+) < 0 \]

\[ \Rightarrow d_u = 0.8125 \]

\[ d_R = \frac{d_L + d_u}{2} = 0.75 + 0.8125 = 0.78125 \]

\[ \left| e_R \right| = \left| \frac{0.78125 - 0.8125}{0.78125} \right| = 0.04 \]
Problem 3 (30 pts)

Use the Newton-Raphson Method to find the value of x where the linear function \( y = x + \frac{1}{2} \) and the trigonometric function \( y = \cos(x) \) are equal. Start with an initial guess of \( x_0 = 0 \). Fill in the table below and round the answers for \( x_i \) and \( x_{i+1} \) to 4 places after the decimal point. Use the rounded answers for \( x_i \) to calculate \( x_{i+1} \). Stop when there is no change in \( x_i \) (rounded to 4 places after the decimal point).

<table>
<thead>
<tr>
<th>i</th>
<th>( x_i )</th>
<th>( f(x_i) )</th>
<th>( f'(x_i) )</th>
<th>( x_{i+1} )</th>
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<tbody>
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<td>1</td>
<td>0.5</td>
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</table>
\[ \begin{align*}
X &= X + \frac{1}{2} \quad \Rightarrow \quad Y = \cos X \\
\Rightarrow \quad X + \frac{1}{2} &= \cos X \\
\Rightarrow \quad f(x) &= X - \cos X + 0.5 \\
f'(x) &= 1 + \sin X \\
X_{i+1} &= X_i - \frac{f(x)}{f'(x)} \\
X_0 &= 0
\end{align*} \]

\[ i = 0, \quad X_1 = X_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{0 - 1 + 0.5}{1 + 0} = 0.5 \]

\[ i = 1, \quad X_2 = X_1 - \frac{f(x_1)}{f'(x_1)} = 0.5 - \frac{0.5 - \cos(0.5) + 0.5}{1 + \sin(0.5)} = 0.4173 \]

\[ i = 2, \quad X_3 = X_2 - \frac{f(x_2)}{f'(x_2)} = 0.4173 - \frac{0.4173 - \cos(0.4173) + 0.5}{1 + \sin(0.4173)} = 0.4151 \]

\[ i = 3, \quad X_4 = X_3 - \frac{f(x_3)}{f'(x_3)} = 0.4151 - \frac{0.4151 - \cos(0.4151) + 0.5}{1 + \sin(0.4151)} = 0.4151 \]