Problem 1 (25 pts)

A spline fit through the points (1,0), (2,4), and (3,9) is required. Find the coefficients $b_1$, $c_1$, $a_2$, $b_2$, and $c_2$ in the function

$$f(x) = \begin{cases} 
\frac{b_1 x + c_1}{a_2 x^2 + b_1 x + c_2} & \text{if } 1 \leq x \leq 2 \\
\frac{b_1 + c_1}{2} & \text{if } 2 \leq x \leq 3
\end{cases}$$

At $x = 1$,

$b_1 + c_1 = 0$
$x = 3$,

$9a_2 + 3b_2 + c_2 = 9$
$x = 2$,

$2b_1 + c_1 = 4$
$4a_2 + 2b_2 + c_2 = 4$
$4a_2 + b_2 = 4$

Slopes are equal: $b_1 = 2a_2(2) + b_2$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 4 & 2 & 1 & 4 \\ 9 & 3 & 1 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow \frac{1}{4}c_2 = 0, c_2 = 0$

$\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow b_2 = 0$

$\Rightarrow a_2 = 1$

Ans. $b_1 = 4$
$c_1 = 4$
$a_2 = 1$
$b_2 = 0$
$c_2 = 0$
Problem 2 (25 pts)

Find the equation of the Newton Divided Difference polynomial which passes through the points (−1, 0), (0, 2), (1, 0) and (2, 0). Assume \( x_0 = -1, \ x_1 = 0, \ x_2 = 1, \) and \( x_3 = 2. \) Show that the polynomial reduces to the product of the three linear factors \((x+1)(x-1)(x-2)\) since the roots are located at \( x = -1, 1, \) and \( 2. \)

\[
\begin{array}{cccc}
\hline
i & x_i & f(x_i) & \alpha_i & \alpha_i^2 & \alpha_i^3 \\
0 & -1 & 0 & 2 & -2 & 1 \\
1 & 0 & 2 & -2 & 1 \\
2 & 1 & 0 & 0 & 0 \\
3 & 2 & 0 & 0 & 0 \\
\hline
\end{array}
\]

\[
\begin{align*}
\frac{f(x_1, x_0)}{x_1 - x_0} &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2 - 0}{-1 - (-1)} = 2 \\
\frac{f(x_2, x_1)}{x_2 - x_1} &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0 - 2}{1 - 0} = -2 \\
\frac{f(x_3, x_2)}{x_3 - x_2} &= \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{0 - 0}{2 - 1} = 0 \\
\frac{f(x_2, x_1, x_0)}{x_2 - x_0} &= \frac{f(x_2, x_1) - f(x_1, x_0)}{x_2 - x_0} = \frac{-2 - 2}{-1 - (-1)} = -2 \\
\frac{f(x_3, x_2, x_1)}{x_3 - x_1} &= \frac{f(x_3, x_2) - f(x_2, x_1)}{x_3 - x_1} = \frac{0 - (-2)}{1 - 0} = 2 \\
\frac{f(x_3, x_2, x_1, x_0)}{x_3 - x_0} &= \frac{f(x_3, x_2, x_1) - f(x_2, x_1, x_0)}{x_3 - x_0} = \frac{1 - (-2)}{2 - (-1)} = 1 \\
\end{align*}
\]

\textbf{Ans.} \( f_3(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2) \)

\[
\begin{align*}
&= x + (x+1)x + (x+1)x(x-1) \\
&= (x+1)[x - 2x + x^2 - x] = (x+1)(x^2 - 3x + 2) \\
&= (x+1)(x-1)(x-2)
\end{align*}
\]
Problem 3 (25 pts)

The exponential \( f(x) = e^x \) is to be approximated by a linear function. Evaluate the exponential at \( x=0,1,2,3 \) and find the equation of the least squares line based on the four data points generated. Find SSE, SSR, SST and the coefficient of determination \( r^2 \). Prepare a table to show how the coefficients of the normal equations were obtained. Prepare a second table to show how SSE, SSR, and SST were calculated.

\[
\begin{array}{c|c|c|c|c}
  x_i & \hat{y}_i & x_i^2 & x_i y_i \\
\hline
  0 & 1 & 0 & 0 \\
  1 & 2.718 & 1 & 2.718 \\
  2 & 7.389 & 4 & 14.778 \\
  3 & 20.086 & 9 & 60.258 \\
\end{array}
\]

\( \hat{y} = \beta_0 + \beta_1 x \)

\[
\begin{array}{c|c|c|c|c|c}
  x_i & y_i & (y_i - \bar{y})^2 & x_i^2 & (x_i - \bar{x})^2 \\
\hline
  0 & 1 & 11.42213 & 0 & 0 \\
  1 & 2.718 & 25.806 & 1 & 1 \\
  2 & 7.389 & 1.167 & 4 & 4 \\
  3 & 20.086 & 150.995 & 9 & 9 \\
\end{array}
\]

\( \sum x_i = 6 \) \quad \sum y_i = 31.193 \quad \sum x_i^2 = 14 \quad \sum x_i y_i = 77.754 

\( \bar{y} = 7.798 \)

\[
\begin{array}{c|c|c|c|c|c}
  6 & 31.193 & \hline
  77.754 & 14 \\
\hline
  4 & 6 \\
  6 & 14 \\
\end{array}
\]

\( \beta_0 = \frac{31.193 \cdot 6 - 77.754 \cdot 14}{6 \cdot 14 - 4 \cdot 6} = -1.491 \)

\( \beta_1 = 6.193 \)

\( \hat{y} = -1.491 + 6.193 x \)

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
  x_i & y_i & (y_i - \bar{y})^2 & x_i^2 & (x_i - \bar{x})^2 & \hat{y}_i & (y_i - \hat{y}_i)^2 & \hat{y}_i^2 & (y_i - \hat{y}_i)^2 \\
\hline
  0 & 1 & 11.42213 & 0 & 0 & -1.491 & 6.205 & 2.205 & 0 \\
  1 & 2.718 & 25.806 & 1 & 1 & 1.957 & 3.970 & 15.806 & 0.205 \\
  2 & 7.389 & 1.167 & 4 & 4 & 7.798 & 10.955 & 59.254 & 0.167 \\
  3 & 20.086 & 150.995 & 9 & 9 & 18.307 & 17.448 & 308.432 & 0.023 \\
\end{array}
\]

\( \text{SST} = 223.181 \quad \text{SSE} = 31.421 \)

\( \text{SSR} = \text{SST} - \text{SSE} = 223.181 - 31.421 = 191.760 \)

\( r^2 = \frac{\text{SSR}}{\text{SST}} = \frac{191.760}{223.181} = 0.859 \)

Ans. \( \text{SSE} = 31.421, \text{SSR} = 191.760, \text{SST} = 223.181, r^2 = 0.859 \)
Problem 4  (25 pts)

Find the equation of the Lagrange interpolating polynomial through the points in the table below. Estimate the value of the function when $x = 2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>5</td>
<td>125</td>
</tr>
</tbody>
</table>

$$f_3(x) = \sum_{i=0}^{3} L_i(x) \cdot f(x_i) = L_1(x) + 27L_2(x) + 125L_3(x)$$

$$L_1(x) = \frac{x(x-3)(x-5)}{(1-0)(1-3)(1-5)} = \frac{x(x-3)(x-5)}{8}$$

$$L_2(x) = \frac{x(x-1)(x-5)}{(3-0)(3-1)(3-5)} = \frac{-x(x-1)(x-5)}{12}$$

$$L_3(x) = \frac{x(x-1)(x-3)}{(5-0)(5-1)(5-3)} = \frac{x(x-1)(x-3)}{40}$$

$$f_3(2) = \frac{1}{8} x(-1)(-3) - \frac{27}{12} - \frac{x(-1)(-3)}{8} + \frac{125}{4} \cdot 2(1)(-1)$$

$$= \frac{6}{8} + \frac{27}{2} - \frac{25}{4} = 8$$

Ans. $f_3(x) = \frac{1}{8} x(x-3)(x-5) - \frac{27}{12} x(x-1)(x-5) + \frac{125}{8} x(x-1)(x-3)$

$$f_3(2) = 8$$
Problem 2  (25 pts)

Find the equation of the Newton Divided Difference polynomial which passes through the points (-1,0), (0,2), (1,0) and (2,0). Assume \( x_0 = -1, \ x_1 = 0, \ x_2 = 1, \) and \( x_3 = 2. \) Show that the polynomial reduces to the product of the three linear factors \((x+1)(x-1)(x-2)\) since the roots are located at \( x = -1, 1 \) and 2.

Ans. \( f_3(x) = \)
Problem 3 (25 pts)

The exponential $f(x) = e^x$ is to be approximated by a linear function. Evaluate the exponential at $x=0,1,2,3$ and find the equation of the least squares line based on the four data points generated. Find SSE, SSR, SST and the coefficient of determination $r^2$. Prepare a table to show how the coefficients of the normal equations were obtained. Prepare a second table to show how SSE, SSR, and SST were calculated. Use three places after the decimal point in all intermediate calculations.
Problem 4  (25 pts)

Find the equation of the Lagrange interpolating polynomial through the points in the table below. Estimate the value of the function when \( x = 2 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>1</td>
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</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>5</td>
<td>125</td>
</tr>
</tbody>
</table>

\[
\text{Ans. } f_3(x) = \\

f_3(2) =
\]