SP 94
EGN 3420 FINAL Name ______________________

DO <u>ANY</u> 4 PROBLEMS
(Check the box next to the problem if you want it graded)
Show All Work! Use Only Methods Discussed In Class!

☐ Problem 1 (25 pts)

Estimate $\int_0^{\pi/2} \sin x \, dx$

A) Using Trapezoidal Integration with 10 equally spaced intervals.

B) Using Simpsons 1/3 Rule with 10 equally spaced intervals.

Fill in the following table to help with the calculations. Round all calculations and calculator display numbers to 4 places after the decimal.

i	x_i	f_i
0	0.0000	0.0000
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

Ans. I (Trapezoidal) = ________, I (Simpson's) = ________

□ Problem 2 (25 pts)

The following integral is to be approximated using Trapezoidal Integration.

$$I = \int_0^4 \left[\frac{-1}{12}(x-2)^4 + 2x^2 \right] dx$$

Let I_n represent the approximation to I when the interval (0,4) is subdivided into n equal sub-intervals. How many sub-intervals must there be so that the error $| I - I_n |$ is at most $\frac{10^{-4}}{3}$.

Ans. n = __________

□ Problem 3 (25 pts)

A quadratic spline is to be fit thru the following points: (0,1), (1,2), (3,3), (4,2).

The spline is given by

$$f(x) = \begin{cases} b_1x + c_1 & 0 \le x \le 1 \\ a_2x^2 + b_2x + c_2 & 1 \le x \le 3 \\ a_3x^2 + b_3x + c_3 & 3 \le x \le 4 \end{cases}$$

A system of 8 equations in the 8 unknowns $b_1,c_1,a_2,b_2,c_2,a_3,b_3,c_3$ has been generated, i.e. $A\underline{x} = \underline{b}$. The vectors $\underline{x}$ and $\underline{b}$ are given below. Find the coefficient matrix A.

$$\underline{x} = \begin{matrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{matrix} \qquad \underline{b} = \begin{matrix} 2 \\ 2 \\ 3 \\ 3 \\ 1 \\ 2 \\ 0 \\ 0 \end{matrix}$$

☐ Problem 4 (25 pts)

An unknown function generated the following data points:

(0,0), (1,1.6487), (3,13.4451)

A) Find the Newton Divided Difference interpolating polynomial $f_2(x)$.

B) An additional data point (2,5.4366) is available. Estimate the error in $f_2(2.5)$.

C) The data points were generated from the function $f(x) = xe^{x/2}$. Find the true error in $f_2(2.5)$.

Ans. A) $f_2(x) =$ ____________________

B) Estimate of error in $f_2(2.5) =$ __________

C) True error in $f_2(2.5) =$ __________

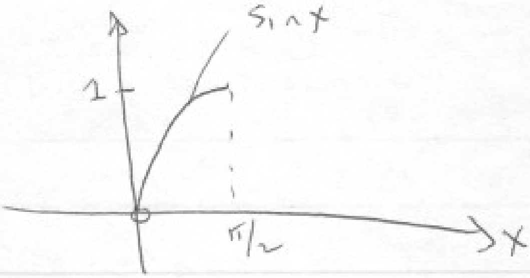
□ Problem 5 (25 pts)

Solve the following system of equations by using a sequence of elementary row operations on the augmented matrix until it has been reduced to its Echelon form.

$$\begin{array}{ccccccccc}
w & + & x & + & y & - & z & = & 1\\
2w & - & x & + & 3y & - & 2z & = & -2\\
w & & & + & y & - & z & = & 0\\
 & & 2x & + & y & & & & 1
\end{array}$$

Ans. $w =$ __________, $x =$ __________, $y =$ __________, $z =$ __________

① $\displaystyle\int_0^{\pi/2} \sin x \, dx$



i	x_i	f_i
0	0	0.0000
1	$\pi/20$	0.1564
2	$\pi/10$	0.3090
3	$3\pi/20$	0.4540
4	$\pi/5$	0.5878
5	$\pi/4$	0.7071
6	$3\pi/10$	0.8090
7	$7\pi/20$	0.8910
8	$2\pi/5$	0.9511
9	$9\pi/20$	0.9877
10	$\pi/2$	1.0000

A) $$I_T = \Delta x \left[\frac{f_0 + f_{10}}{2} + (f_1 + f_2 + \cdots + f_9) \right]$$

$$= \frac{\pi}{20} \left[\frac{0+1}{2} + (0.1564 + \cdots + 0.9877) \right]$$

$$= 0.9979$$

B) $$I_S = \frac{\Delta x}{3}\left[f_0 + 4f_1 + 2f_2 + \cdots + 2f_8 + 4f_9 + f_{10} \right]$$

$$= \frac{\Delta x}{3}\left[(f_0 + f_{10}) + 4(f_1 + f_3 + f_5 + f_7 + f_9) + 2(f_2 + f_4 + f_6 + f_8) \right]$$

$$= \frac{\pi/20}{3}\left[(0+1) + 4(0.1564 + \cdots + 0.9877) + 2(0.3090 + \cdots + 0.9511) \right]$$

$= [illegible]\ 0.9999$

C) $$\int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = 0 - (-1) = 1$$

TRAPEZOIDAL: $e_T = \left| \dfrac{1 - \cancel{1.0043}}{1} \right| \times 100 =$ 0.[illegible] %

SIMPSONS: $e_T = \left| \dfrac{1 - \cancel{1.0042}}{1} \right| \times 100 =$ 0.[illegible] %

(2) $$I = \int_0^4 \left[\frac{-1}{12}(x-2)^4 + 2x^2 \right] dx$$

Let I_n = Approximation to I using Trap. Integ.

$$I - I_n = \frac{-1}{12} f''(\xi)(b-a)h^2 \qquad a \le \xi \le b$$

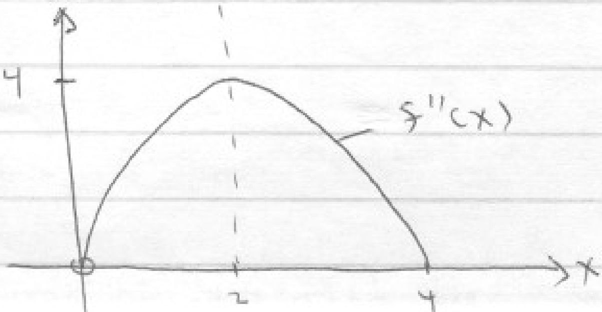
$$\Rightarrow |I - I_n| = \frac{1}{12} |f''(\xi)| (b-a) h^2$$

Maximum Possible $|I - I_n| = \frac{1}{12} \left| \max_{0 \le x \le 4} f''(x) \right| (b-a) h^2$

$$f(x) = \frac{-1}{12}(x-2)^4 + 2x^2$$

$$f'(x) = \frac{-4}{12}(x-2)^3 + 4x$$

$$f''(x) = -(x-2)^2 + 4$$



$$\max_{0 \le x \le 4} f''(x) = 4$$

$\Rightarrow$ Max Possible $|I - I_n| = \frac{1}{12}(4)4h^2 = \frac{4h^2}{3}$

$$\Rightarrow \quad \frac{4h^2}{3} = \frac{10^{-4}}{3}$$

$$4h^2 = 10^{-4}$$

$$h = \frac{10^{-2}}{2} = \frac{b-a}{n} = \frac{4-0}{n}$$

$$\Rightarrow \quad n = \frac{4 \times 2}{10^{-2}}$$

Ans. $n = 800$ intervals

③ Fit a quadratic spline thru the pts $(0,1)$, $(1,2)$, $(3,3)$ & $(4,2)$

$$f(x) = \begin{cases} b_1 x + c_1 & 0 \le x \le 1 \\ a_2 x^2 + b_2 x + c_2 & 1 \le x \le 3 \\ a_3 x^2 + b_3 x + c_3 & 3 \le x \le 4 \end{cases}$$

Interior Pts:

$x=1$, $b_1 + c_1 = 2$

$a_2 + b_2 + c_2 = 2$

$x=3$, $9a_2 + 3b_2 + c_2 = 3$

$9a_3 + 3b_3 + c_3 = 3$

End Pts:

$x=0$, $c_1 = 1$

$x=4$, $16a_3 + 4b_3 + c_3 = 2$

Interior Pt Derivatives:

$x=1$, $b_1 = 2a_2 + b_2$

$x=3$, $2a_2(3) + b_2 = 2a_3(3) + b_3$

Putting in the form $A\underline{x} = \underline{b}$

where $\underline{x} = \begin{bmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix}$

$$A = \begin{array}{c} \begin{array}{cccccccc} b_1 & c_1 & a_2 & b_2 & c_2 & a_3 & b_3 & c_3 \end{array} \\ \begin{bmatrix} 1 & 1 & & & & & & \\ & & 1 & 1 & 1 & & & \\ & & 9 & 3 & 1 & & & \\ & & & & & 9 & 3 & \cancel{3}1 \\ & 1 & & & & & & \\ & & & & & 16 & 4 & 1 \\ 1 & & -2 & -1 & & & & \\ & & 6 & \cancel{3}1 & & -6 & \overset{-1}{\cancel{3}} & \end{bmatrix} \end{array}, \quad \underline{b} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 3 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

(4) Evaluate $f(x) = xe^{x/2}$ at $x = 0, 1, 3$

and find

A) NDD 2^{nd} order polynomial thru the 3 pts
B) Add a pt at $[2, f(2)]$ and estimate the error in the 2^{nd} order polynomial at $x = 2.5$
C) Compare the estimate of the error with the true error, i.e. find the true error.

$x_0 = 0$ $\quad$ $x_1 = 1$ $\quad$ $x_2 = 3$

$f(x_0) = 0$ $\quad$ $f(x_1) = 1.6487$ $\quad$ $f(x_2) = 13.4451$

i	x_i	$f(x_i)$	Δ	Δ^2	Δ^3
0	0	0	1.6487	1.4165	0.3469
1	1	1.6487	5.8982	2.1103	
2	3	13.4451	8.0085		
3	2	5.4366			

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1.6487 - 0}{1 - 0} = 1.6487$$

$$f[x_2, x_1] = \frac{13.4451 - 1.6487}{3 - 1} = 5.8982$$

$$f[x_3, x_2] = \frac{5.4366 - 13.4451}{2 - 3} = 8.0085$$

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{5.8982 - 1.6487}{3 - 0} = 1.4165$$

$$f[x_3, x_2, x_1] = \frac{8.0085 - 5.8982}{2 - 1} = 2.1103$$

$$f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0}$$

$$= \frac{2.1103 - 1.4165}{2 - 0} = 0.3469$$

$$\Rightarrow f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= 0 + 1.6487x + 1.4165\,x(x-1)$$

B) $f_3(x) - f_2(x) = b_3(x - x_0)(x - x_1)(x - x_2)$

$$= 0.3469\,x(x-1)(x-3)$$

$$f_3(2.5) - f_2(2.5) = 0.3469(2.5)(1.5)(-0.5) = -0.6504$$

(Estimate of Error)

c) $f(2.5) = 2.5e^{2.5/2} = 8.7259$

$f_2(2.5) = 1.6487(2.5) + 1.4165(2.5)(1.5) = 9.4336$

True Error $E_T = f(2.5) - f_2(2.5) = 8.7259 - 9.4336 = -0.7077$

(5) $w = 1 + z$

$x = 1$

$y = -1$

$z =$ arbitrary

$w + x + y + z = (1 + z) + 1 - 1 + z = 1 + 2z$

$\Rightarrow \quad w + x + y - z = 1 \qquad (A)$

$2w - x + 3y = 2(1+z) - 1 + 3(-1) = 2z - 2$

$\Rightarrow \quad 2w - x + 3y - 2z = -2 \qquad (B)$

$w + y = (1 + z) - 1 = z$

$\Rightarrow \quad w + y - z = 0 \qquad (C)$

$2x + y + 3z = 2 - 1 + 3z = 1 + 3z$

$\Rightarrow \quad 2x + y = 1$

Solve:

$$\begin{aligned}
w + x + y - z &= 1 \\
2w - x + 3y - 2z &= -2 \\
w + y - z &= 0 \\
2x + y &= 1
\end{aligned}$$

$$
(A \,\vdots\, \underline{b}) = \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 2 & -1 & 3 & -2 & -2 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{array}\right] \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & -3 & 1 & 0 & -4 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 \end{array}\right]
$$

$$
\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & -3 & 1 & 0 & -4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right]
$$

$$
\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}
$$

$\Rightarrow$ $y = -1$

$x = 1$

$w + x + y - z = 1$

$w + 1 - 1 - z = 1$

$w = 1 + z$

Soln: $w = 1 + z$

$x = 1$

$y = -1$

$z =$ arbitrary