Fa 94

EGN 3420 — Exam 2 — ~~Midterm -~~ Part I — Name ________________

Problem 1 (20 pts)

Given the function $f(x) = e^{-x}$, evaluate the n^{th} order truncated Taylor Series $f_n(x)$ for n=0,1,2,3,4,5, at x=1/2 using a base pt of $x_0=0$. Fill in the table below. Round all numerical calculations to 4 places after the decimal point.

n	$f_n(1/2)$	$f(1/2)$	e_T
0	1.0000	0.6065	-0.6487
1			
2			
3			
4			
5			

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Problem 2 (20 pts)

Use the Newton-Raphson Method to estimate the root of $f(x) = x^2 - e^{-x}$. Start the iterations with an initial guess of $x_0 = 0$. Fill in the table below. The iterated values x_i, $i=1,2,3$ should be rounded to 4 places after the decimal point, and the magnitude of the approximate relative error (as a percent) to 2 places after the decimal point.

i	x_i	$\lvert e_A \rvert$, %
0	0	
1		
2		
3	0.7038	4.15

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Problem 3 (20 pts)

Use the Bisection Method to estimate the root of $f(x) = x^3 - 1.5x^2 + 2x - 3$ starting with an initial bracket of $x_l=0$ and $x_u=10$. Fill in the table below, stopping when $|f(x_r)| \leq 0.5$. All calculations should be rounded to 4 places after the decimal point.

x_l	x_u	x_r	$f(x_l)$	$f(x_r)$
0	10			

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Problem 4 (20 pts)

An unknown function f(x) generated the following data points:

(0,3), (1,3), (1.5,3.25), (2, 1.6667)

The objective is to estimate the function at x=1.25 by using a 2nd order Newton Divided Difference interpolating polynomial. That is,

$$f_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$$

A) Choose the appropriate 3 points from the given data set and fill in the table below of divided diferences. Round all calculations to 4 places after the decimal point.

i	x_i	$f(x_i)$	Δ	Δ^2	Δ^3
0	1	3	0.5	-3.6667	-2
1	1.5	3.25	-3.1667	-1.6667	
2	2	1.6667	-0.6667		
3	0	3			

B) Use $f_2(1.25)$ as the estimate of f(1.25)

Ans. $f_2(1.25)$ = 3.3542

C) Use the additional data point to find $f_3(x)$.

Ans. $f_3(x)$ = $3 + 0.5(x-1) - 3.6667(x-1)(x-1.5) - 2(x-1)(x-1.5)(x-2)$

D) Estimate the error in $f_2(1.25)$ as $R_2 = f_3(1.25) - f_2(1.25)$.

Ans. R_2 = -0.0938

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Problem 5 (20 pts)

Consider the three points from the function $f(x) = \ln x$.

$x_0=1, \quad f(x_0) = f(1) = 0$
$x_1=e, \quad f(x_1) = f(e) = 1$
$x_2=e^2, \quad f(x_2) = f(e^2) = 2$

The Lagrange Interpolating polynomial thru the data points is

$$f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

A) Find $L_1(x)$ and $L_2(x)$.

Ans. $L_1(x) = \dfrac{(x-1)(x-e^2)}{(e-1)(e-e^2)}$

$L_2(x) = \dfrac{(x-1)(x-e)}{(e^2-1)(e^2-e)}$

B) Use $f_2(x)$ to estimate $f(x)$ when $x=2$.

Ans. $f_2(2) =$ 0.6233

C) Find the true error, E_T.

Ans. $E_T =$ 0.0698

Work Area

(1)

$$f(x) = e^{-x}$$

$$f'(x) = -e^{-x}$$

$$f''(x) = e^{-x}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^n e^{-x}$$

$$f_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-x_0)^k$$

For $x_0 = 0$, $x = 1/2$

$$f_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} e^{-x_0} (x-x_0)^k$$

$$f_n(1/2) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} (1/2)^k$$

$$= \sum_{k=0}^{n} \frac{1}{k!}(-1/2)^k = 1 - \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{6}\left(\frac{1}{2}\right)^3 + \cdots + \frac{1}{n!}\left(-\frac{1}{2}\right)^n$$

$n=0$, $f_0(1/2) = 1$, $e_T = \dfrac{f(1/2) - f_0(1/2)}{f(1/2)} = \dfrac{e^{-1/2} - 1}{e^{-1/2}} = -0.6487$

$n=1$, $f_1(1/2) = 1 - \frac{1}{2} = \frac{1}{2}$, $e_T = \dfrac{f(1/2) - f_1(1/2)}{f(1/2)} = \dfrac{e^{-1/2} - 1/2}{e^{-1/2}} = 0.1756$

$n=2$, $f_2(1/2) = 1 - \frac{1}{2} + \frac{1}{8} = \cancel{\frac{3}{4}}\ \frac{5}{8}$, $e_T = \dfrac{f(1/2) - f_2(1/2)}{f(1/2)} = \dfrac{e^{-1/2} - \cancel{\frac{3}{4}}\frac{5}{8}}{e^{-1/2}} = \cancel{-0.2365}$ -0.0305

$n = 3, \quad f_3(1/2) = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} = \cancel{\frac{5}{8}} \;\; \frac{29}{48}$

$$e_T = \frac{f(1/2) - f_3(1/2)}{f(1/2)} = \frac{e^{-1/2} - 29/8}{e^{-1/2}} = \cancel{-0.0305} \;\; 0.003897$$

$n = 4, \quad f_4(1/2) = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \frac{11}{16}$

$$e_T = \frac{f(1/2) - f_4(1/2)}{f(1/2)} = \frac{e^{-1/2} - 11/16}{e^{-1/2}} = -0.1335$$

$n = 5, \quad f_5(1/2) = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} = \frac{21}{32}$

$$e_T = \frac{f(1/2) - f_5(1/2)}{f(1/2)} = \frac{e^{-1/2} - 21/32}{e^{-1/2}} = -0.0820$$

(2) $f(x) = x^2 - e^{-x}$

$f'(x) = 2x + e^{-x}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^2 - e^{-x_i}}{2x_i + e^{-x_i}}$$

$$= \frac{2x_i^2 + x_i e^{-x_i} - x_i^2 + e^{-x_i}}{2x_i + e^{-x_i}}$$

$$x_{i+1} = \frac{x_i^2 + (1+x_i)e^{-x_i}}{2x_i + e^{-x_i}}$$

$x_0 = 0$

$$x_1 = \frac{x_0^2 + (1+x_0)e^{-x_0}}{2x_0 + e^{-x_0}} = \frac{0+1}{0+1} = 1$$

$$|e_A| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100 = \left|\frac{1-0}{1}\right| \times 100 = 100\,\%$$

$$x_2 = \frac{x_1^2 + (1+x_1)e^{-x_1}}{2x_1 + e^{-x_1}} = \frac{1 + 2e^{-1}}{2 + e^{-1}} = 0.7330$$

$$|e_A| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100 = \left|\frac{0.7330 - 1}{0.7330}\right| \times 100 = 36.43\,\%$$

$$x_3 = \frac{x_2^2 + (1+x_2)e^{-x_2}}{2x_2 + e^{-x_2}} = \frac{(0.7330)^2 + 1.7330\, e^{-0.7330}}{2(0.7330) + e^{-0.7330}} = 0.7038$$

$$|\epsilon_A| = \left|\frac{x_3 - x_2}{x_3}\right| \times 100 = \left|\frac{0.7038 - 0.7330}{0.7038}\right| \times 100 = 4.15\%$$

③ $f(x) = x^3 - 1.5x^2 + 2x - 3$

$x_L = 0$

$x_U = 10$

x_L	x_U	x_R	$f(x_L)$	$f(x_R)$
0	10	5	-3	94.5
0	5	2.5	-3	8.25
0	2.5	1.25	-3	-0.8906
1.25	2.5	1.875	-0.8906	2.0684
1.25	1.875	1.5625	-0.8906	0.2776

$$x_R = \frac{x_L + x_U}{2} = \frac{0+10}{2} = 5$$

(4)

x	0	1	3/2	2
$f(x)$	3	3	13/4	5/3

Estimate $f(1.25)$ from a 2nd NDD interpolating poly.

$$f_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$$

$x_0 = 1$

$x_1 = 1.5$

$x_2 = 2$

i	x_i	$f(x_i)$	Δ	Δ^2	Δ^3
0	1	3	0.5	-3.6667	-2
1	1.5	13/4	-3.1667	-1.6667	
2	2	5/3	-0.6667		
3	0	3			

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\frac{13}{4} - 3}{\frac{3}{2} - 1} = 0.5$$

$$f[x_2, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\frac{5}{3} - \frac{13}{4}}{2 - \frac{3}{2}} = -\frac{19}{6} = -3.1667$$

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{-\frac{19}{6} - \frac{1}{2}}{2-1} = \frac{-22}{6} = -3.6667$$

$f_2(x) = 3 + 0.5(x-1) - 3.6667(x-1)(x-1.5)$

$\Rightarrow f_2(1.25) = 3 + 0.5(0.25) - 3.6667(0.25)(-0.25)$

$= 3.3542$

To estimate error in $f_2(1.25)$ use additional data pt at $(0,3)$, i.e. $x_3 = 0$ & $f(x_3) = 3$

$$f[x_3, x_2] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{3 - 5/3}{0 - 2} = -0.6667$$

$$f[x_3, x_2, x_1] = \frac{f[x_3, x_2] - f[x_2, x_1]}{x_3 - x_1} = \frac{-\frac{2}{3} - \left(-\frac{19}{6}\right)}{0 - 3/2} = -\frac{5}{3}$$

$$f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0} = \frac{-\frac{5}{3} - \left(-\frac{22}{6}\right)}{0 - 1} = -2$$

$f_3(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2)$

$= f_2(x) + b_3(x-x_0)(x-x_1)(x-x_2)$

Error Estimate, $R_2 = f_3(1.25) - f_2(1.25)$

$= b_3(1.25-1)(1.25-1.5)(1.25-2)$, $b_3 = -2$

$= -2(0.25)(-0.25)(-0.75)$

$= -0.09375$

(5) $f(x) = \ln x$

x_i	$f(x_i)$
$x_0 - 1$	0
$x_1 - e$	1
$x_2 - e^2$	2

$$f_2(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$
$$= L_1(x) + 2L_2(x)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-e^2)}{(e-1)(e-e^2)} = \frac{-1}{e(e-1)^2}(x-1)(x-e^2)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-e)}{(e^2-1)(e^2-e)} = \frac{1}{e(e-1)^2(e+1)}(x-1)(x-e)$$

$$\Rightarrow f_2(x) = \frac{1}{e(e-1)^2}\left[-(x-1)(x-e^2) + \frac{2}{e+1}(x-1)(x-e)\right]$$

$$f_2(2) = \frac{1}{e(e-1)^2}\left[-(1)(2-e^2) + \frac{2}{e+1}(1)(2-e)\right]$$

$$= \frac{(e^2-2)(e+1) + 2(2-e)}{e(e-1)^2(e+1)}$$

$$= \frac{e^3 + e^2 - 4e + 2}{e(e-1)(e^2-1)}$$

$$= 0.6233$$

$$E_T = f(2) - f_2(2) = \ln 2 - 0.6233$$
$$= 0.6931 - 0.6233 = 0.0698$$