Problem 1 (30 pts)

A continuous integrator with initial condition $x(0) = 1$ is shown below.

\[ x(0) = 1 \]

\[ u(t) = e^{-2t} \]

\[ \int \rightarrow x(t) \]

Fill in the table below where $x(t_n)$ is the exact value of the integrator output at time $t_n$. Choose $T = 0.05$ for each integrator. Round all answers to 4 places after the decimal point.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_n = nT$</th>
<th>$x_A(n)$ Explicit Euler</th>
<th>$x_A(n)$ Trapezoidal</th>
<th>$x(t_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A second order system is modeled by the differential equation
\[
\frac{d^2 w}{dt^2} + \frac{dw}{dt} + 2w = 3\frac{d^2 u}{dt^2}.
\]

The initial conditions \( w(0) = \frac{dw}{dt}(0) = 0 \).

a) Draw a simulation diagram for the system.

b) Find matrices \( A, B, C, D \) in the state variable model form
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

The single output is \( y = w \).

c) The input \( u(t) = t, \ t \geq 0 \). Use explicit Euler integration with step size \( T = 0.1 \) to find \( y_A(1), y_A(2) \) and \( y_A(3) \). Round answers to 4 places after the decimal point.

Hint: First find
\[
\begin{bmatrix}
x_{1,A}(1) \\
x_{2,A}(1)
\end{bmatrix}, \begin{bmatrix}
x_{1,A}(2) \\
x_{2,A}(2)
\end{bmatrix}, \begin{bmatrix}
x_{1,A}(3) \\
x_{2,A}(3)
\end{bmatrix}
\]
Problem 3 (35 pts)

An exponential population growth model

\[ \frac{dP}{dt} = -kP, \ (k > 0) \]

is to be simulated in order to approximate the population \( P(t) \) for a period of time. The difference equation for \( P_A(n) \) intended to approximate \( P(t) \) is

\[ P_A(n+1) + \alpha P_A(n) = 0 \]

a) Find an expression for \( \alpha \) in terms of \( k \) and the step size \( T \) using
   i) Explicit Euler
   ii) Implicit Euler

b) Evaluate \( \alpha \) for each integrator when \( k = 0.1 \) and \( T = 1 \) yr. Round answers to 4 places after the decimal point.

c) Fill in the Table below when \( P(0) = 5 \) million. Round all answers in millions to 4 places after the decimal point.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t_n = nT )</th>
<th>( P_A(n), ) Explicit Euler</th>
<th>( P_A(n), ) Implicit Euler</th>
<th>( P(t_n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

d) The exact solution is given by \( P(t) = P(0)e^{-kt}, \ t \geq 0 \). Fill in the last column of the table.