Problem 1 (30 pts)
A continuous integrator with initial condition is shown below.

\[ u(t) = e^{-2t} \]

\[ x(0) = 1 \]

\[ x(t) \]

Fill in the table below. Choose \( T = 0.05 \) for each integrator. Round all answers to 4 places after the decimal point.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_i(n) ) Explicit Euler</th>
<th>( x_i(n) ) Trapezoidal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Problem 2 (35 pts)

A second order system is modeled by the differential equation

\[ \frac{d^2w}{dt^2} + \frac{dw}{dt} + 2w = \frac{d^2u}{dt^2}. \]

The initial conditions \( w(0) = \frac{dw}{dt}(0) = 0. \)

a) Draw a simulation diagram for the system.

b) Find matrices \( A, B, C, D \) in the state variable model form

\[ \dot{x} = Ax + Bu \\
\]

\[ y = Cx + Du \]

The single output is \( y = w. \)

c) The input \( u(t) = t, \ t \geq 0. \) Use explicit Euler integration with step size \( T = 0.1 \) to find \( y_d(1), y_d(2) \) and \( y_d(3). \) Round answers to 4 places after the decimal point.

Hint: First find \( x_d(1), x_d(2) \) and \( x_d(3). \)
Problem 3 (35 pts)

An exponential population growth model

\[ \frac{dP}{dt} = -kP, \quad (k > 0) \]

is to be simulated in order to approximate the population \( P(t) \) for a period of time. The difference equation for \( P_A(n) \) intended to approximate \( P(t) \) is

\[ P_A(n+1) + \alpha P_A(n) = 0 \]

a) Find an expression for \( \alpha \) in terms of \( k \) and the step size \( T \) using
   i) Implicit Euler
   ii) Trapezoidal
   iii) Improved Euler

b) Evaluate \( \alpha \) for each integrator and round answers to 6 places after decimal point.

c) Fill in the Table comparing the three numerical integrators and the exact solution. \( P(0) = 5 \) million and \( T = 2 \) yr. Round all answers in millions to 6 places after the decimal point. For the exact, enter \( P(T) \) and \( P(2T) \).

<table>
<thead>
<tr>
<th></th>
<th>( P_A(0) )</th>
<th>( P_A(1) )</th>
<th>( P_A(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implicit Euler</td>
<td>5.000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trapezoidal</td>
<td>5.000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Improved Euler</td>
<td>5.000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td>5.000000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>