Sample Question#1

1. Prove that the following are equivalent
   a) $S$ is an infinite recursive (decidable) set.
   b) $S$ is the range of a monotonically increasing total recursive function.
   Note: $f$ is monotonically increasing means that $\forall x \ f(x+1) > f(x)$.

DONE in class and notes.
2. Let A and B be re sets. For each of the following, either prove that the set is re, or give a counterexample that results in some known non-re set.

Let A be semi decided by $f_A$ and B by $f_B$

a) $A \cup B$: must be re as it is semi-decided by 
$$f_{A \cup B}(x) = \exists t \ [ \text{stp}(f_A, x, t) \ or \ \text{stp}(f_B, x, t) ]$$

b) $A \cap B$: must be re as it is semi-decided by 
$$f_{A \cap B}(x) = \exists t \ [ \text{stp}(f_A, x, t) \ and \ \text{stp}(f_B, x, t) ]$$

c) $\neg A$: can be non-re. If $\neg A$ is always re, then all re are recursive as any set that is re and whose complement is re is decidable. However, $A = K$ is a non-rec, re set and so $\neg A$ is not re.
Sample Question#3

3. Present a demonstration that the *even* function is primitive recursive.
   
   \[
   \text{even}(x) = 1 \text{ if } x \text{ is even} \\
   \text{even}(x) = 0 \text{ if } x \text{ is odd}
   \]

   You may assume only that the base functions are prf and that prf’s are closed under a finite number of applications of composition and primitive recursion.

   DONE in class.
Sample Question#4

4. Given that the predicate **STP** and the function **VALUE** are prf’s, show that we can semi-decide

\{ f \mid \varphi_f \text{ evaluates to 0 for some input} \}

This can be shown re by the predicate

\{ f \mid \exists<x,t> [\text{stp}(f,x,t) \&\& \text{value}(f,x,t) = 0] \}
5. Let $S$ be an re (recursively enumerable), non-recursive set, and $T$ be re, non-empty, possibly recursive set. Let $E = \{ z \mid z = x + y, \text{where } x \in S \text{ and } y \in T \}$.

(a) Can $E$ be non re? **No** as we can let $S$ and $T$ be semi-decided by $f_S$ and $f_T$, resp., $E$ is then semi-dec. by $f_E (z) = \exists <x,y,t> \ [\text{stp}(f_S, x, t) \&\& \text{stp}(f_T, y, t) \&\& (z = \text{value}(f_S, x, t) \ast \text{value}(f_T, y, t))]$

(b) Can $E$ be re non-recursive? **Yes**, just let $T = \{0\}$, then $E = S$ which is known to be re, non-rec.

(c) Can $E$ be recursive? **Yes**, let $T = \aleph$, then $E = \{ x \mid x \geq \min (S) \}$ which is a co-finite set and hence rec.
Sample Question #6

6. Assuming \textbf{TOTAL} is undecidable, use reduction to show the undecidability of
\[ \text{Incr} = \{ f \mid \forall x \ \varphi_f(x+1) > \varphi_f(x) \} \]
Let \( f \) be arb.
Define \( G_f(x) = \varphi_f(x) - \varphi_f(x) + x \)
\( f \in \text{TOTAL} \) iff \( \forall x \varphi_f(x) \downarrow \) iff \( \forall x \ G_f(x) \downarrow \) iff \( \forall x \ \varphi_f(x) - \varphi_f(x) + x = x \) iff \( G_f \in \text{Incr} \)
Sample Question#7

7. Let \( \text{Incr} = \{ f \mid \forall x, \varphi_f(x+1) > \varphi_f(x) \} \).
Let \( \text{TOT} = \{ f \mid \forall x, \varphi_f(x) \downarrow \} \).
Prove that \( \text{Incr} \equiv_m \text{TOT} \). Note Q#6 starts this one.

Let \( f \) be arb.
Define \( G_f(x) = \exists t [\text{stp}(f,x,t) \&\& \text{stp}(f,x+1,t) \&\& (\text{value}(f,x+1,t) > \text{value}(f,x,t))] \)
\( f \in \text{Incr} \) iff \( \forall x \varphi_f(x+1) > \varphi_f(x) \) iff
\( \forall x G_f(x) \downarrow \) iff \( G_f \in \text{TOT} \)
8. Let \( \text{Incr} = \{ f \mid \forall x \varphi_f(x+1) > \varphi_f(x) \} \). Use Rice’s theorem to show \( \text{Incr} \) is not recursive.

Non-Trivial as

\( C_0(x) = 0 \notin \text{Incr}; \ S(x) = x+1 \in \text{Incr} \)

Let \( f,g \) be arb. Such that \( \forall x \varphi_f(x) = \varphi_g(x) \)

\( f \in \text{Incr} \iff \forall x \varphi_f(x+1) > \varphi_f(x) \) iff

\( \forall x \varphi_g(x+1) > \varphi_g(x) \) iff \( g \in \text{Incr} \)
9. Let $S$ be a recursive (decidable set), what can we say about the complexity (recursive, re non-recursive, non-re) of $T$, where $T \subset S$?

Nothing. Just let $S = \mathbb{N}$, then $T$ could be any subset of $\mathbb{N}$. There are an uncountable number of such subsets and some are clearly in each of the categories above.
Sample Question#10

10. Define the pairing function $<x,y>$ and its two inverses $<z>_1$ and $<z>_2$, where if $z = <x,y>$, then $x = <z>_1$ and $y = <z>_2$.

Right out of Notes.
Sample Question#11

11. Assume $A \leq_m B$ and $B \leq_m C$. 
Prove $A \leq_m C$.

Done in class
12. Let $P = \{ f \mid \exists x [ \text{STP}(f, x, x) ] \}$. Why does Rice’s theorem not tell us anything about the undecidability of $P$?

This is not an I/O property as we can have implementations of $C_0$ that are efficient and satisfy $P$ and others that do not.