Let set $A$ be non-empty recursive, $B$ be re non-recursive and $C$ be non-re. Using the terminology (REC) recursive, (RE) non-recursive recursively enumerable, (NR) non-re, categorize each set below, saying whether or not the set can be of the given category and justifying each answer. You may assume, for any set $S$, the existence of comparably hard sets $S_E = \{ 2x | x \in S \}$ and $S_D = \{ 2x+1 | x \in S \}$. The following is a sample of the kind of answer I require:

Sample.) $A \cap C = \{ x | x \in A \text{ and } x \in C \}$

**REC:** Yes. If $A = \{0\}$ then $A \cap C = \emptyset$ or $\{0\}$, each of which is in REC.

**RE:** Yes. Let $A = \aleph_E = \{ 2x | x \in \aleph \}$; let $C = TOT_D \cup \text{HALT}_E$ then $A \cap C = \text{HALT}_E$ which is in RE

**NR:** Yes. If $A = \aleph$ then $A \cap C = \aleph$, which is in NR.

a.) $B - A = \{ x | x \in B \text{ and } x \notin A \}$ // Set difference

**REC:** Yes. Let $A = \aleph$, $B = \text{HALT}$, then $B - A = \emptyset$, which is in REC

**RE:** Yes. Let $A = \{0\}$, $B = \text{HALT}$, then $B - A = \text{HALT} - \{0\}$, which is in RE

**NR:** This is not possible. To see this, consider that $B - A = B \cap \sim A$. Since $A$ is in REC, then so is $\sim A$. A semi-decision procedure for $B - A$ can be constructed by first seeing the chosen number, call it $x$, is in $\sim A$. If it is not then answer “NO.” If it is then run the semi-decision procedure for $B$. If it ever answers “YES,” produce the answer “YES.” Formally, if $\chi_A$ decides $A$ and $g_B$ semi-decides $B$, the $g_{B-A}(x) = (1 - \chi_A(x)) \times g_B(x)$ semi-decide $B - A$.

b.) $A^* B = \{ x^* y | x \in A \text{ and } y \in B \}$ // Multiplication

**REC:** Let $A = \{0\}$, $B = \text{HALT}$, then $A^* B = \{0\}$, which is in REC

**RE:** Let $A = \{1\}$, $B = \text{HALT}$, then $A^* B = \text{HALT}$, which is in RE

**NR:** This is not possible. To see this, we need just show that we can semi-decide $A^* B$. By definition $x \in A^* B$ iff $\exists a \in A, b \in B$ such that $x = a^* b$. But, the rules of multiplication of natural numbers then implies we can limit our search to values of $a,b$ such $0 \leq a,b \leq x$, so we have bounds on each value. To take advantage of this, let’s get rid of 0 first. If $x = 0$, then $x$ is in $A^* B$, if 0 is in $A$ (quick check) or 0 is in $B$ (semi-decidable). Thus, we can semi-decide this case. For all other cases, just search for each $a$, $1 \leq a \leq x$ such that $a$ divides $x$. Include each such $x/a$ in a list called Check. Now run the enumerating function for $B$ to see if any of these items show up. If any does, answer “yes.” This provides a semi-decision procedure.

c.) $A \cup C = \{ x | x \in A \text{ or } x \in C \}$ // Set union

**REC:** Let $A = \aleph$, $C = \sim \text{HALT}$, then $A \cup C = \aleph$, which is in REC

**RE:** Let $A = \aleph_D$, $B = \text{HALT}_E \cup \sim \text{HALT}_D$, then $A \cup C = \text{HALT}_E \cup \aleph_D$, which is in RE

**NR:** Let $A = \{0\}$, $C = \sim \text{HALT}$, then $A \cup C = \sim \text{HALT} \cup \{0\}$, which is in NR