COT 6410: Pipeline Scheduling

Michael Gabilondo
Overview

1. Create a Formal Problem from a Real-world Problem
2. Investigate Complexity of Defined Problem
3. Redefine the Problem
4. NP-Completeness for Revised Problem
5. References
I chose a real-world problem: Instruction Scheduling on a Pipeline with precedence constraints between pipeline stages of jobs. or, Pipeline-Scheduling for short.
I chose a real-world problem: **Instruction Scheduling on a Pipeline with precedence constraints between pipeline stages of jobs**. or, **Pipeline-Scheduling** for short.

It’s this problem, from your Computer Architecture course:

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The IBM System/360, which implemented Tomasulo’s algorithm, is a notable example of a dynamic scheduling processor [Hennessy, “Computer Architecture”].
A different view of the pipeline chart

Here, the P1 P2 P3 P4 P5 are pipeline stages: IF ID EX MEM WB.
A different view of the pipeline chart

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- Each instruction is a **JOB**.
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Each instruction is a JOB.

Each Instruction/Job has m tasks, the number of pipeline stages. Here, m = 5.
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This appears to be a shop-scheduling problem.
Defining a shop-scheduling problem

- Starting point is **flow-shop scheduling problem**, which is NP-Complete; informal description
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- Task $i + 1$ of a job $j$ cannot start until task $i$ has completed; i.e., the tasks are ordered
- Tasks have a length, $l(t)$; if a task $t$ on processor $i$ is scheduled on $\sigma_i(t)$, then nothing else can be scheduled on that processor until $\sigma_i(t) + l(t)$
QUESTION: Can the tasks be scheduled on the processors such that the above constraints are obeyed and the finishing time of the last task is less than D?
How is the pipeline problem different from the m-Machine flow-shop problem?

- All tasks are unit length, i.e., \( l(t_i(j)) = 1 \), for all \( 1 \leq i \leq m, 1 \leq j \leq n \).
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- This makes our problem a type of no-wait flow-shop, i.e., m-machine no-wait flow-shop.
- A partial order \( < \) is defined over the tasks. \( t < t' \) means that task \( t' \) cannot begin until task \( t \) finishes because, for example, \( t \) produces data that is required by \( t' \).
Define our problem formally

Pipeline Scheduling
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**Pipeline Scheduling**

**INSTANCE:**
- A set $J$ of jobs, a number $m$ of processors, each job $j$ has tasks $t_1(j), t_2(j), \ldots, t_m(j)$, each task having a length $l(t_i(j)) = 1$.
- A partial order $\prec$ over the tasks, indicating precedence constraints between tasks.
- An integer $D$, the deadline for the last task.
Define our problem formally

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  - A partial order $\prec$ over the tasks, indicating precedence constraints between tasks.
  - An integer $D$, the deadline for the last task.

- **QUESTION:**
  - Is there a schedule with the finishing time of the last task less than $D$, and the schedule is a **no-wait flow-shop schedule** and if $t_{i'}(j') < t_i(j)$, then $\sigma_{i'}(t_{i'}(j')) < \sigma_i(t_i(j))$?
Investigate the Complexity of Created Problem

- Is pipeline scheduling NP-Complete?
Is pipeline scheduling NP-Complete?

This is formulated as a scheduling (rather than one-processor sequencing) problem, so I attempted to use 3-Machine No-Wait Flow-Shop, but no luck.
Is pipeline scheduling NP-Complete?

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I found another problem which represents the pipeline problem more closely.
MINIMUM PRECEDENCE CONSTRAINED SEQUENCING WITH DELAYS

- An NP-Complete problem
MINIMUM PRECEDENCE CONSTRAINED SEQUENCING WITH DELAYS

- An NP-Complete problem
- **INSTANCE**: Set T of tasks, a directed acyclic graph $G = (T, E)$ defining precedence constraints for the tasks, a positive integer D, and for each task an integer delay $0 \leq d(t) \leq D$. 

![Diagram](image.png)
**QUESTION**: Is there a one-processor schedule $S$ for $T$ that obeys the precedence constraints and the delays, and the maximum $S(t) \leq D$? The schedule $S$ is an injective function $S : T \rightarrow Z^+$ such that, for each edge

$$\langle t_i, t_j \rangle \in E, S(t_j) - S(t_i) > d(t_i) \iff S(t_j) \geq S(t_i) + d(t_i) + 1$$
Proving NP-Completeness

- I tried to find a polynomial transformation from sequencing problem to the pipeline problem
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In pipeling scheduling, if $m$ is exponential, then that instance will contain sets with exponential number of elements.
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In precedence constrained sequencing, the integers $d(t)$ and $D$ can be exponential in magnitude, and the instance will still be polynomial.

In pipelining scheduling, if $m$ is exponential, then that instance will contain sets with exponential number of elements.

My transformation created sets of elements with $D$ or $d(t)$ elements, so it is not a polynomial transformation.

I modified pipeline scheduling so that sets of tasks were not part of the problem, and modified the precedence constraints; I will show how this new formulation is very similar to the old problem.
Redefining Pipeline Scheduling

What is the effect of the precedence constraints between tasks on the starting times of the first task of each job?

- Job 4 cannot start until 2 units after job 2 starts.
- Task 2 requires task 4.
Observation. If $t_k(j') < t_i(j)$ and $k \geq i$, then the starting time of the first task of job $j$, $\sigma_1(j) \geq \sigma_1(j') + k - i + 1$, if $k \geq i$. 
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If $t_k(j') < t_i(j)$ and $k < i$, then the first task of job $j$ must be scheduled at least one time unit after job $t_1(j')$ is scheduled.
**Observation.** If \( t_k(j') < t_i(j) \) and \( k \geq i \), then the starting time of the first task of job \( j \), \( \sigma_1(j) \geq \sigma_1(j') + k - i + 1 \), if \( k \geq i \).

If \( t_k(j') < t_i(j) \) and \( k < i \), then the first task task of job \( j \) must scheduled at least one time unit after job \( t_1(j') \) is scheduled.

Also, the maximum “delay” a precedence constraint can have is \( k - i + 1 \leq m \).
Redefining Pipeline Scheduling

- We don’t have to worry about tasks and can just represent jobs. We can create a DAG to represent $\prec$, and put weights on the edges using a weight function $W$: 
Redefining Pipeline Scheduling

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$$\langle t_i, t_j \rangle \in E \iff S(t_j) \geq S(t_i) + W(\langle t_i, t_j \rangle) + 1$$

In this case, we put an edge between Job 4 and Job 2 with weight 2.
Definition of Pipeline Scheduling Revised

- Pipeline Sequencing problem
Definition of Pipeline Scheduling Revised

- **Pipeline Sequencing problem**

- **INSTANCE:** A set $I$ of instructions: $\{I_1, I_2, \ldots, I_n\}$, $m$ pipeline stages, $G = (I, E)$, a weighted directed acyclic graph (DAG), a weight function $W : E \rightarrow \{1, 2, 3, \ldots, m - 1\}$. 
Definition of Pipeline Scheduling Revised

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- **QUESTION:** Is there a one-processor schedule $\sigma : I \rightarrow \mathbb{Z}_0^+$, such that $\sigma$ is one-to-one,
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$\langle I_j, I_k \rangle \in E \Leftrightarrow \sigma(I_k) \geq \sigma(I_j) + W(\langle I_j, I_k \rangle) + 1$
Definition of Pipeline Scheduling Revised

- **Pipeline Sequencing problem**
- **INSTANCE**: A set \( I \) of instructions: \( \{I_1, I_2, ..., I_n\} \), \( m \) pipeline stages, \( G = (I, E) \), a weighted directed acyclic graph (DAG), a weight function \( W : E \rightarrow \{1, 2, 3, ..., m-1\} \).
- **QUESTION**: Is there a one-processor schedule \( \sigma : I \rightarrow \mathbb{Z}_0^+ \), such that \( \sigma \) is one-to-one,
  - \( \langle I_j, I_k \rangle \in E \iff \sigma(I_k) \geq \sigma(I_j) + W(\langle I_j, I_k \rangle) + 1 \)
  - For the maximum \( \sigma(I) \), does \( \sigma(I) + (m - 1) \leq D \)?
Proof that Pipeline Sequencing is NP-Complete

- Accept an instance of MINIMUM PRECEDENCE CONSTRAINED SEQUENCING WITH DELAYS.
Proof that Pipeline Sequencing is NP-Complete

- Accept an instance of MINIMUM PRECEDENCE CONSTRAINED SEQUENCING WITH DELAYS.

- INSTANCE: Set $T$ of tasks, a directed acyclic graph $G = (T, E)$ defining precedence constraints for the tasks, a positive integer $D$, and for each task an integer delay $0 \leq d(t) \leq D$. 
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- Accept an instance of **MINIMUM PRECEDENCE CONSTRAINED SEQUENCING WITH DELAYS**.

- **INSTANCE**: Set $T$ of tasks, a directed acyclic graph $G = (T, E)$ defining precedence constraints for the tasks, a positive integer $D$, and for each task an integer delay $0 \leq d(t) \leq D$.

- **QUESTION**: Is there a one-processor schedule $S$ for $T$ that obeys the precedence constraints and the delays, and the maximum $S(t) \leq D$? The schedule $S$ is an injective function $S : T \rightarrow \mathbb{Z}^+$ such that, for each edge $\langle t_i, t_j \rangle \in E, S(t_j) - S(t_i) > d(t_i) \iff S(t_j) \geq S(t_i) + d(t_i) + 1$
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- Create an instance of PIPELINE SEQUENCING, using the existing instance of MINIMUM PRECEDENCE CONSTRAINED SEQUENCING WITH DELAYS
Proof that Pipeline Sequencing is NP-Complete

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Proof that Pipeline Sequencing is NP-Complete

Create an instance of **PIPELINE SEQUENCING**, using the existing instance of **MINIMUM PRECEDENCE CONSTRAINED SEQUENCING WITH DELAYS**

- \( I \leftarrow T \),
- \( E' \leftarrow E \)
Proof that Pipeline Sequencing is NP-Complete

Create a Formal Problem from a Real-world Problem
Investigate Complexity of Defined Problem
Redefine the Problem
NP-Completeness for Revised Problem
References

Create an instance of pipeline sequencing, using the existing instance of minimum precedence constrained sequencing with delays

- I ← T,
- E′ ← E
- For each edge \( \langle t_i, t_j \rangle \in E \)
Proof that Pipeline Sequencing is NP-Complete

Create an instance of pipeline sequencing, using the existing instance of minimum precedence constrained sequencing with delays

\[ I \leftarrow T, \]
\[ E' \leftarrow E \]
For each edge \( \langle t_i, t_j \rangle \in E \)

\[ W(\langle t_i, t_j \rangle) \leftarrow d(t_i) \]
Proof that Pipeline Sequencing is NP-Complete

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  - \[ m \leftarrow \max_{t \in T} \{d(t)\} + 1 \]
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Proof that Pipeline Sequencing is NP-Complete

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- Solve pipeline sequencing and return the answer
Proof that Pipeline Sequencing is NP-Complete

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- \( I \leftarrow T \),
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- For each edge \( \langle t_i, t_j \rangle \in E \)
  - \( W(\langle t_i, t_j \rangle) \leftarrow d(t_i) \)
  - \( m \leftarrow \max\{d(t)\} + 1 \)
  - \( D' \leftarrow D + (m - 1) \)

- Solve pipeline sequencing and return the answer
- The time complexity is \( O(T + E) \); this is a polynomial transformation.
Proof that Pipeline Sequencing is NP-Complete

Notice that if $\langle t_i, t_j \rangle \in E$, then $S(t_j) \geq S(t_i) + d(t_i) + 1$. The equivalent condition can hold in pipeline sequencing by adding that edge to $E'$ and and setting the weight on that edge to $d(t_i)$. 
Proof that Pipeline Sequencing is NP-Complete

- Notice that if \((t_i, t_j) \in E\), then \(S(t_j) \geq S(t_i) + d(t_i) + 1\). The equivalent condition can hold in PIPELINE SEQUENCING by adding that edge to \(E'\) and and setting the weight on that edge to \(d(t_i)\).

- The respective instances and constraints of the two problems have been constructed to be equivalent:
Proof that Pipeline Sequencing is NP-Complete

Notice that if \( \langle t_i, t_j \rangle \in E \), then \( S(t_j) \geq S(t_i) + d(t_i) + 1 \). The equivalent condition can hold in pipeline sequencing by adding that edge to \( E' \) and setting the weight on that edge to \( d(t_i) \).

The respective instances and constraints of the two problems have been constructed to be equivalent:

If the sequencing with delays instance is true and the last job is scheduled at time \( t \), then the last job of pipeline sequencing will also be scheduled at time \( t \). The pipeline sequencing instance adds \( m - 1 \) time units to the finishing time \( t \), so we set \( D' \) to \( D + (m - 1) \).
Proof that Pipeline Sequencing is NP-Complete

If the pipeline sequencing instance is true, then it has a schedule that meets its constraints; if the last job of the pipeline sequencing instance is scheduled at time t, then so will the corresponding job in sequencing with delays also be scheduled at time t. In pipeline sequencing, the problem ends $m + 1$ time units after the last job; this is why $m + 1$ time units were added to the deadline D when constructing the pipeline sequencing instance.
Proof that Pipeline Sequencing is NP-Complete

- If the **pipeline sequencing** instance is true, then it has a schedule that meets its constraints; if the last job of the **pipeline sequencing** instance is scheduled at time t, then so will the corresponding job in **sequencing with delays** also be scheduled at time t. In **pipeline sequencing**, the problem ends $m + 1$ time units after the last job; this is why $m + 1$ time units were added to the deadline D when constructing the **pipeline sequencing** instance.

- Thus, the constructed instance is true if and only if the base instance is true.
Proof that Pipeline Sequencing is NP-Complete

- If the **pipeline sequencing** instance is true, then it has a schedule that meets its constraints; if the last job of the **pipeline sequencing** instance is scheduled at time $t$, then so will the corresponding job in **sequencing with delays** also be scheduled at time $t$. In **pipeline sequencing**, the problem ends $m + 1$ time units after the last job; this is why $m + 1$ time units were added to the deadline $D$ when constructing the **pipeline sequencing** instance.

- Thus, the constructed instance is true if and only if the base instance is true.

- Now, once I show Pipeline Sequencing is in NP, then that proves it is NP-Complete.
Pipeline Sequencing is in NP

Is pipeline sequencing in NP? An oracle can provide the starting time for each of the instructions, and an algorithm simply needs to check the DAG to see if the schedule is valid. For each edge in the DAG, the algorithm simply needs to check if the starting time of the dependant task is large enough and comes after the first task; it also needs to check the deadline is met by simply checking the starting time of the last instruction. So the problem is NP.
Conclusion

- This has show that the real-life problem of **Instruction Scheduling on a Pipeline with precedence constraints between pipeline stages of jobs** is NP-Hard for an **unbounded** number of processors. In practice, this result does not apply because most machines have a small number of pipeline stages.

- We were unable to show that the **pipeline scheduling problem with sets of m tasks** was NP-Complete because our **polynomial transformation** was creating instances where m was exponential. We have not shown it is not NP-complete, however; we don’t know.
But, I do not think it is NP-Complete, because the paper by Hennesy showed that the sequencing with delays problem was NP-Complete for unbounded $m$. They had to introduce extra constraints into the problem to get a bound on $m$. 

References