Image De-noising Problem

Liyue Zhao
Outline

- Problem
  - Description
  - Definition
- Proof
Problem
Problem

- Description
  - Each image $I$ consists of $m$ by $n$ binary pixels
  - Each pixel $I(i,j)$ corresponding to a boolean variable $X_{i,j} = \{1, 0\}$
  - Each two adjacent pixels are connected with an edge
Problem

- Goal
  - Given a binary noise image $I'$
  - Generate a new binary image $I$ that eliminates the noise in $I'$
Problem

• Definition 1
  • *The observation* is the value of node of known noise image, and a *instant* is a function which assigns a value to each variable in the new image
Problem

- Definition 2
  - A \textit{clique} $Q_k(X_k)$ is a complete sub-graph with a set of variables in clique
Problem

- **Definition 3**
  - A **clique table (CT)** $T$ for a variable $X$ with a set of variables in **clique** $Q$ is a function that maps each **instant** of $Q$ to a binary value

<table>
<thead>
<tr>
<th>$\phi_{Q_k}$</th>
<th>$X_{i-1}$</th>
<th>$X_i$</th>
<th>$X_{i+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_{i-1}$</td>
<td>$x_i$</td>
<td>$x_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>$\overline{x}_{i-1}$</td>
<td>$x_i$</td>
<td>$x_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>$x_{i-1}$</td>
<td>$\overline{x}_i$</td>
<td>$x_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>$x_{i-1}$</td>
<td>$x_i$</td>
<td>$\overline{x}_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>$x_{i-1}$</td>
<td>$\overline{x}_i$</td>
<td>$\overline{x}_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>$\overline{x}_{i-1}$</td>
<td>$x_i$</td>
<td>$\overline{x}_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>$\overline{x}_{i-1}$</td>
<td>$\overline{x}_i$</td>
<td>$x_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>$\overline{x}_{i-1}$</td>
<td>$\overline{x}_i$</td>
<td>$\overline{x}_{i+1}$</td>
</tr>
</tbody>
</table>
Problem

- Simple rules
  - If the observations of a clique have same value, the variables in the clique can’t against all of them
  - If the observation of a clique have different value, the variables in the clique can’t follow all of them
**Problem**

- **Example**
  - Assume dark node is 0 and light node is 1, since variables can’t follow the value of observations, \((x_{i-1} \land \bar{x}_i \land x_{i+1}) = 0\)

\[
\begin{array}{|c|c|c|c|}
\hline
\phi_{C_k} & X_{i-1} & X_i & X_{i+1} \\
\hline
1 & x_{i-1} & x_i & x_{i+1} \\
1 & \bar{x}_{i-1} & x_i & x_{i+1} \\
0 & x_{i-1} & \bar{x}_i & x_{i+1} \\
1 & x_{i-1} & x_i & \bar{x}_{i+1} \\
1 & x_{i-1} & \bar{x}_i & \bar{x}_{i+1} \\
1 & \bar{x}_{i-1} & x_i & \bar{x}_{i+1} \\
1 & \bar{x}_{i-1} & \bar{x}_i & x_{i+1} \\
1 & \bar{x}_{i-1} & \bar{x}_i & \bar{x}_{i+1} \\
\hline
\end{array}
\]
Problem

- Therefore, as the example with the function of clique $k$ could be represented as the

\[ \phi_{Q_k} = (x_{i-1} \land x_i \land x_{i+1}) \lor (x_{i-1} \land x_i \land \overline{x}_{i+1}) \lor (\overline{x}_{i-1} \land x_i \land x_{i+1}) \lor (x_{i-1} \land \overline{x}_i \land \overline{x}_{i+1}) \lor (\overline{x}_{i-1} \land \overline{x}_i \land \overline{x}_{i+1}) \lor (\overline{x}_{i-1} \land \overline{x}_i \land x_{i+1}) \lor (\overline{x}_{i-1} \land x_i \land \overline{x}_{i+1}) \lor (x_{i-1} \land \overline{x}_i \land x_{i+1}) \]
A Markov network is a pair \((G,Q)\) where \(G\) is an undirected graph whose nodes are variables and \(Q\) is a set which consists of the clique of each variable.

\[
\text{score}(G,Q) = \prod_{k=1}^{K} \phi_{Q_k}(Q_k)
\]
Proof
Formal definition: IDP

- Given: A Markov network \((G, Q)\) and an instant of variables \(X\)
- Question Does there exist an assignment to \(X\) so that

\[
\prod_{k=1}^{K} \phi_{Q_k} (Q_k) = 1
\]
Formal definition: 3-SAT

- Given: A set $U$ of boolean variables and a collection $C_i \subseteq U \cup \overline{U}$, and $|C_i| = 3$
- Question: Does there exist an assignment to $U$ so that all clauses are true?
Proof

- IDP $\in$ NP
  
  for $i=1$ to $K$
    
    for $j=1$ to 7 (the formula number of clique table)
      
      check if the variables in clique $i$ is satisfying one of the formulas
    
    end
  
  end

end
Proof

- Trick
  - Just check the inverse value of the zero formula in the clique table

- Example
  - As shown in previous table, the zero formula is \( (x_{i-1} \land \overline{x}_i \land x_{i+1}) \)
  - Check the variables value of inverse formula \( (\overline{x}_{i-1} \land x_i \land \overline{x}_{i+1}) \)
  - If one(or more) of the element of inverse formula is true, then

\[
\phi_{Q_k} = (x_{i-1} \land x_i \land x_{i+1}) \lor (x_{i-1} \land x_i \land \overline{x}_{i+1}) \lor (\overline{x}_{i-1} \land x_i \land x_{i+1}) \lor \\
(\overline{x}_{i-1} \land x_i \land \overline{x}_{i+1}) \lor (\overline{x}_{i-1} \land \overline{x}_i \land x_{i+1}) \lor \\
(\overline{x}_{i-1} \land \overline{x}_i \land \overline{x}_{i+1})
\]

\[
= 1
\]
Proof

- Input of IDP
  - $X =$ boolean variables
  - $Q =$ cliques
  - $T =$ table of each clique

- Input of 3SAT
  - $U =$ $X \cup \bar{X}$
  - $C =$ $\{C_1, \ldots, C_K\}$, where $C_k$ is the inverse of zero formula
Proof

- Yes(IDP) → Yes(3SAT)
  - If IDP is true, which means each clique function $\phi_{Q_k} = 1$, the corresponding clause $C_k$ is true.
  - Clearly 3SAT is true

- Yes(3SAT) → Yes(IDP)
  - If 3SAT is true, for each clause, one of the formulas in corresponding table T except zero formula must be true
  - IDP is true
Conclusion

\[ \text{IDP} \propto 3\text{SAT} \]
References

• James D. Park, Using Weighted MAX-SAT Engines to Solve MPE, in *AAAI*, 2002