Frame Building Problem

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Outline

1 Basic Problem
   - Description
   - Examples

2 Proof
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2. Proof
Informal Description

- Need frames to build a greenhouse.
- We have some boards we can use.
- Boards can be cut to make smaller frames.
- Want to minimize extra wood needed.
Formal Definition: \textit{FBP}

**Given**

A set $F$ of frames $\{f_i\}$ and a set $B$ of boards $\{b_i\} \cup \{E\}$, where $E$ is the length of the extra board.

**Question**

Does there exist an assignment of every frame to a board such that the sum of the lengths of frames assigned to a board is no greater than the length of the board?
Formal-er Definition: FBP

**Given**

A set $F$ of frames $\{f_i\}$ and a set $B$ of boards $\{b_i\} \cup \{E\}$, where $E$ is the length of the extra board.

**Question**

Does there exist a total mapping $M : F \rightarrow B$ such that

$$\forall b_k \in B : \sum_{\substack{j \in M^{-1}(k) \cap F}} f_j \leq b_k$$
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## Examples

### Example 1

\[ B = \{20, 10\} \cup \{5\} \]
\[ F = \{10, 5, 4, 3, 3, 3\} \]

### Example 2

\[ B = \{20, 10\} \cup \{5\} \]
\[ F = \{10, 5, 4, 4, 3, 3, 3\} \]

### Example 3

\[ B = \{20, 10\} \cup \{5\} \]
\[ F = \{10, 8, 5, 4, 3, 3, 3\} \]
Failed Approaches

- **Multidimensional Knapsack**
  - Similar flavor
  - \((v_i, w_i)\) versus \(f_i\)
  - Partial versus total mapping

- **Zero-One Integer Programming**
  - Define the family of indicator variables \(a_{ij}\) that indicate whether \(f_i\) is cut from \(b_j\).
  - \[\sum_{i} a_{ij} f_i \leq b_j\] subject to \[\sum_{ij} a_{ij} = |F|\].
BinPacking[1]

Given
Finite set $U$ of items, a size $s(u) \in \mathbb{Z}^+$ for each $u \in U$, a positive integer bin capacity $B$, and a positive integer $K$.

Question
Is there a partition of $U$ into disjoint sets $U_1, U_2, \ldots, U_k$ such that the sum of the sizes of the items in each $U_i$ is $B$ or less?
Formal Definition: $FBP_K$

**Given**

A set $F$ of frames $\{f_i\}$ and a set $B$ of boards $\{b_i\} \cup \{E\}$, where $E$ is the length of the extra board, and an integer $K$.

**Question**

Does there exist an assignment of every frame to a board such that the sum of the lengths of frames assigned to a board is no greater than the length of the board, using at most $K$ boards?
Bin Packing $\preceq FBP_K$

- Restrict $FBP_K$ to instances where all boards have the same length.
- $b_1 = b_2 = \cdots = b_{|B|}$
- Boards correspond to bins and frames correspond to items.
$FBP_K \propto FBP$

- Reduction from $FBP_K$ to $FBP$.

**Input to $FBP_K$**
- $F$
- $B$
- $E$
- $K$

**Input to $FBP$**
- $F' = F$
- $B' = \{\text{Largest } K \text{ elements from } B\}$
- $E' = \text{Largest element from } B$
\[ FBP_K \propto FBP \]

- \( \text{Yes}(FBP) \rightarrow \text{Yes}(FBP_K) \)
  - Clearly, this is true, since the created instance of \( FBP \) contains exactly \( K \) boards.

- \( \text{Yes}(FBP_K) \rightarrow \text{Yes}(FBP) \)
  - If the solution to \( FBP_K \) uses only elements of \( B' \), then this is clearly true.
  - Else, \( \exists b_i, b_j \ni b_i \text{ is used, } b_j \text{ is not used, } b_i \leq b_j \text{ and } b_j \in B' \).
  - Then every frame assigned to \( b_i \) can be assigned to \( b_j \).
  - By induction, this transforms any solution to a Yes-instance of \( FBP_K \) to a Yes-instance of \( FBP \).
BinPacking $\sim FBP_K \sim FBP$