COT 6410 Spring 2023 **Midterm** **Name: KEY**
Raw Score ___ / 60 **Grade:________**

6 **1.** In each case below, consider **R1** to be Regular, **R2** to be finite, and **L1** and **L2** to be non-regular CFLs. Fill in the three columns with **Y** or **N**, indicating what kind of language **L** can be. No proofs are required. Read ⊆ as "contained in and may equal."
Put **Y** in all that are possible and **N** in all that are not.

Definition of L	Regular?	CFL, non-Regular?	Not even a CFL?
$L = L1 \cap R2$	*Y*	*N*	*N*
$L = L1 - L2$	*Y*	*Y*	*Y*
$L = \Sigma^* - R1$	*Y*	*N*	*N*
$L \subseteq L1$	*Y*	*Y*	*Y*

6 **2.** Choosing from among **(D) decidable**, **(U) undecidable**, categorize each of the following decision problems. No proofs are required. **L** is a language over **Σ**.

Problem / Language Class	Regular	Context Free	Context Sensitive	Phrase Structured
L contains L^2 ?	*D*	*U*	*U*	*U*
L contains Σ ?	*D*	*D*	*D*	*D*
|L| is cofinite ?	*D*	*D*	*U*	*U*

5 **3.** Prove that any class of languages, ***C***, closed under union, concatenation, intersection with regular languages, homomorphism and substitution (e.g., the Context-Free Languages) is closed under **Cut with Regular Sets**, denoted by the operator ∇, where **L ∈ *C***, **R** is Regular, **L** and **R** are both over the alphabet **Σ**, and
$L \nabla R = \{ w \mid w = xy, \text{ for } x \in \Sigma^+, y \in R, \text{ and either } xy \in L \text{ or } yx \in L \}$.
You may assume substitution **f(a) = {a, a'}**, and homomorphisms **g(a) = a'** and **h(a) = a, h(a') = λ**. Here **a∈Σ** and **a'** is a new character associated with each such **a∈Σ**.
You only need give me the definition of **L ∇ R** in an expression that obeys the above closure properties; you do not need to prove or even justify your expression.

$L \nabla R = h\,(\,f(L) \cap (\,R\,g(\Sigma^+) \cup g(\Sigma^+)\,R\,)$

4 **4.** Specify True (**T**) or False (**F**) for each statement.

Statement	T or F
The Context Sensitive Languages are closed under homomorphism	*F*
The Post Correspondence Problem is undecidable if |Σ| = 1	*F*
The predicate **STP(f,x,t)** is primitive recursive	*T*
If **P** $\le_m$ **Halt** then **P** cannot be decidable	*F*
The **RE** sets are closed under intersection	*T*
Myhill-Nerode proves that every regular language has a minimum state NFA	*F*
If **P** is re and Rice's Theorem applies to **P** then **P** is re-Complete	*T*
The correct terminating traces of a Turing Machine's Computations form a CFL	*F*

4 **5.** Let **P** = **<X = <aba, bb, a>, Y = <bab, b, baa>>**, **Σ = {a,b}**, be an instance of **PCP**. Present a context-free grammar, **G**, associated with this instance of PCP, **P**, such that **$\mathcal{L}$(G)** is ambiguous if and only if there is a solution to **P**. This answer must a specific instance of the general construction that uses the above **X** and **Y**.
Define **G = ({S, X, Y}, Σ, R, S)** where **R is** the set of rules (this is your job):

S → X | Y
S → aba X [1] | bb X [2] | a X [3] | aba [1] | bb [2] | a [3]
S → bab Y [1] | b Y [2] | baa Y [3] | bab [1] | b [2] | baa [3]

12 **6.** Choosing from among **(REC) recursive**, **(RE) re non-recursive, (coRE) co-re non-recursive**, **(NRNC) non-re/non-co-re**, categorize each of the sets in a) through d). Justify your answer by showing some minimal quantification of some known recursive predicate.

a.) B = { f | for all x, φ_f (x) = f }

∀x ∃t [STP(f,x,t) & VALUE(f,x.t) = x] — *NRNC*

b.) C = { f | domain(φ_f) contains the number 0 }

∃t [STP(f,0,t)] — *RE*

c.) D = { <f, x> | φ_f(x) takes at least 10 steps to converge }

~ STP(f, x, 9) — *REC*

d.) A = { f | range(φ_f) does not contain the number 0 }

∀<x, t> [STP(f,x,t) ⇒ VALUE(f,x,t) ≠ 0] — *Co-RE*

2 **7.** Looking back at Question 6, which of these are candidates for using Rice's Theorem to show their unsolvability? Check all for which Rice Theorem might apply.

a) ___*X* **b)** ___*X* **c)** ____ **d)** ___*X*

3 **8.** We wish to prove that, if **S** recursively enumerable and **dom(g_S) = S**, that we can create an algorithm, **f_S**, such that **x** ∈ S iff **f_S(x) = x** and **x** ∉ S iff **f_S(x)↑**.

f_S(x) = *$x * \exists t\ STP(g_S, x, t)$*

6 **9.** Let set **A** be an re non-recursive (undecidable) set that does not contain the value **0**, and let **B** be a **non-empty** recursive (decidable) set.
Consider **C = { z | z = x -- y,** where **x** ∈ **A** and **y** ∈ **B** }. **// -- is limited subtraction (produces max (0, x – y), which is primitive recursive)**
For (a)-(c), either show sets **A** and **B** and the resulting set **C**, such that **C** has the specified property (argue convincingly that it has the correct property) or demonstrate (prove by construction) that this property cannot hold.

a. Can **C** be recursive? Circle **Y** or **N.**

Let A = K and B = $\aleph$
C = K - $\aleph$ = $\aleph$ which is recursive

b. Can **C** be re non-recursive? Circle **Y** or **N.**

Let A = K and B = {0}
C = K – {0} $\equiv_m$ K which is re, non-recursive..

c. Can **C** be non-re? Circle **Y** or **N.**
You may assume **A = range(f_A), B = range(f_B),** for some algorithms **f_A, f_B.**

C can be enumerated by $f_C(<x,y>) = f_A(x) – f_B(y)$
where the minus is limited subtraction.
Thus, C is always re.

10. Define **NonTrivialDomain (NTD) = { f | |Domain(f)| > 1 }.**

2 **a.)** Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of **NTD**.

$\exists \langle x,y,t\rangle\, [\, x \neq y \;\&\; STP(f,x,t) \;\&\; STP(f,y,t)\,]$

4 **b.)** Use Rice's Theorem to prove that **NTD** is undecidable.

Dom(S) = $\aleph$ *and* $|\aleph| > 1$ *so* $S \in NTD$ *(Note all PRFs are in NTD)*

$Dom(\uparrow) = \{\,\}$ *and* $|\{\,\}| = 0$ *so* $\uparrow \notin NTD$

Thus, NTD is non-trivial

Let f and g be indices of functions where dom(f) = dom(g).

$f \in NTD \Leftrightarrow |dom(f)| > 1$ *By definition*

$\Leftrightarrow |dom(g)| > 1$ *As dom(f) = dom(g)*

$\Leftrightarrow g \in NTD$

This shows NTD is undecidable due to Rice's Theorem -- Weak Form based on domains

3 **c.)** Consider **NonTrivialRange (NTR) = { f | |Range(f)| > 1 }.** Show that **NTR** $\leq_m$ **NTD**.

Let f be an arbitrary index. Define $\forall x\; G_f(x) = \exists \langle y,t\rangle\, [\, STP(f,y,t) \;\&\; VALUE(f,y,t) = x\,]$

Note this is $\exists y\, [\, f(y) = x\,]$ *but there is an issue in writing it this way with f a procedure*

$f \in NTR \Leftrightarrow |Range(f)| > 1 \Rightarrow \exists \langle x0, x1\rangle,\; x0 \neq x1$, *where* $x0 \;\&\; x1 \in Range(f)$

$\Rightarrow G_f(x0){\downarrow}$ *and* $G_f(x1){\downarrow},\; x0 \neq x1 \Rightarrow G_f \in NTD$. *Thus,* $f \in NTR \Rightarrow G_f \in NTD$

$f \notin NTR \Leftrightarrow |Range(f)| \leq 1$. *But then either* $\underline{\forall y}\; f(y){\uparrow} \Rightarrow |dom(G_f)| = 0$

Or $\exists x \underline{\forall y}\; f(y){\downarrow} \Rightarrow f(y) = x$ *and so* $G_f(x){\downarrow}$ *but* $G_f(z){\uparrow}$ *if* $z \neq x \Rightarrow |dom(G_f)| = 1$

Thus, $f \notin NTR \Rightarrow G_f \notin NTD$ *and so* $f \in NTR \Leftrightarrow G_f \in NTD$

3 **d.)** Show that **NTD** $\leq_m$ **NTR**.

Let f be an arbitrary index, Define $\forall x\; G_f(x) = x * \exists t\, [\, STP(f,x,t)\,] = f(x) - f(x) + x$

Clearly $G_f(x) = x \Leftrightarrow x \in dom(f)$ *and* $G_f(x){\uparrow}$ *if* $x \notin dom(f)$

Thus, Range(Gf) = dom(Gf) = dom(f) and so their cardinalities are the same.

And so, $f \in NTD \Leftrightarrow G_f \in NTR$