6 1. In each case below, consider R1 to be Regular, R2 to be finite, and L1 and L2 to be non-regular CFLs. Fill in the three columns with Y or N, indicating what kind of language L can be. No proofs are required. Read $\subseteq$ as “contained in and may equal.” Put Y in all that are possible and N in all that are not.

<table>
<thead>
<tr>
<th>Definition of L</th>
<th>Regular?</th>
<th>CFL, non-Regular?</th>
<th>Not even a CFL?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = L_1 \cap R_2$</td>
<td>Y</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>$L = L_1 - L_2$</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>$L = \Sigma^* - R_1$</td>
<td>Y</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>$L \subseteq L_1$</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>

6 2. Choosing from among (D) decidable, (U) undecidable, categorize each of the following decision problems. No proofs are required. L is a language over $\Sigma$.

<table>
<thead>
<tr>
<th>Problem / Language Class</th>
<th>Regular</th>
<th>Context Free</th>
<th>Context Sensitive</th>
<th>Phrase Structured</th>
</tr>
</thead>
<tbody>
<tr>
<td>L contains $L^2$?</td>
<td>D</td>
<td>U</td>
<td>U</td>
<td>U</td>
</tr>
<tr>
<td>L contains $\Sigma$?</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>D</td>
</tr>
<tr>
<td>$</td>
<td>L</td>
<td>$ is cofinite?</td>
<td>D</td>
<td>D</td>
</tr>
</tbody>
</table>

5 3. Prove that any class of languages, C, closed under union, concatenation, intersection with regular languages, homomorphism and substitution (e.g., the Context-Free Languages) is closed under Cut with Regular Sets, denoted by the operator $\triangledown$, where $L \in C$, R is Regular, L and R are both over the alphabet $\Sigma$, and

$L \triangledown R = \{ w \mid w = xy, \text{ for } x \in \Sigma^*, y \in R, \text{ and either } xy \in L \text{ or } yx \in L \}.$

You may assume substitution $f(a) = \{a, a'\}$, and homomorphisms $g(a) = a'$ and $h(a) = a, h(a') = \lambda$. Here $a \in \Sigma$ and $a'$ is a new character associated with each such $a \in \Sigma$.

You only need give me the definition of $L \triangledown R$ in an expression that obeys the above closure properties; you do not need to prove or even justify your expression.

$L \triangledown R = \bigcup h ( f(L) \cap ( R g(\Sigma^*) \cup g(\Sigma^*) R ) )$

4 4. Specify True (T) or False (F) for each statement.

<table>
<thead>
<tr>
<th>Statement</th>
<th>T or F</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Context Sensitive Languages are closed under homomorphism</td>
<td>F</td>
</tr>
<tr>
<td>The Post Correspondence Problem is undecidable if $</td>
<td>\Sigma</td>
</tr>
<tr>
<td>The predicate $\text{STP}(f,x,t)$ is primitive recursive</td>
<td>T</td>
</tr>
<tr>
<td>If $P \leq_m \text{Halt}$ then $P$ cannot be decidable</td>
<td>F</td>
</tr>
<tr>
<td>The RE sets are closed under intersection</td>
<td>T</td>
</tr>
<tr>
<td>Myhill-Nerode proves that every regular language has a minimum state NFA</td>
<td>F</td>
</tr>
<tr>
<td>If $P$ is re and Rice’s Theorem applies to $P$ then $P$ is re-Complete</td>
<td>T</td>
</tr>
<tr>
<td>The correct terminating traces of a Turing Machine’s Computations form a CFL</td>
<td>F</td>
</tr>
</tbody>
</table>
4 5. Let $P = \langle X = \langle \text{aba, bb, a} \rangle, Y = \langle \text{bab, b, baa} \rangle, \Sigma = \{a,b\} \rangle$ be an instance of PCP. Present a context-free grammar, $G$, associated with this instance of PCP, $P$, such that $L(G)$ is ambiguous if and only if there is a solution to $P$. This answer must a specific instance of the general construction that uses the above $X$ and $Y$.

Define $G = (\{S, X, Y\}, \Sigma, R, S)$ where $R$ is the set of rules (this is your job):

\[
\begin{align*}
S & \rightarrow X \mid Y \\
S & \rightarrow \text{aba }X/1 \mid \text{bb }X/2 \mid a X/3 \mid \text{aba }/1 \mid \text{bb }/2 \mid a/3 \\
S & \rightarrow \text{bab }Y/1 \mid \text{b }Y/2 \mid \text{baa }Y/3 \mid \text{bab }/1 \mid \text{b }/2 \mid \text{baa }/3
\end{align*}
\]

12 6. Choosing from among (REC) recursive, (RE) re non-recursive, (coRE) co-re non-recursive, (NRNC) non-re/non-co-re, categorize each of the sets in a) through d). Justify your answer by showing some minimal quantification of some known recursive predicate.

a.) $B = \{ f \mid \text{for all } x, \ \varphi_f(x) = f \}$

\[
\forall x \exists t \ [ \text{STP}(f,x,t) \land \text{VALUE}(f,x,t) = x ] \quad \text{NRNC}
\]

b.) $C = \{ f \mid \text{domain}(\varphi_f) \text{ contains the number 0 } \}$

\[
\exists t \ [ \text{STP}(f,0,t) ] \quad \text{RE}
\]

c.) $D = \{ f, x \mid \varphi_f(x) \text{ takes at least 10 steps to converge } \}$

\[
\neg \text{STP}(f,x,9) \quad \text{REC}
\]

d.) $A = \{ f \mid \text{range}(\varphi_f) \text{ does not contain the number 0 } \}$

\[
\forall <x,t> \ [ \text{STP}(f,x,t) \implies \text{VALUE}(f,x,t) \neq 0 ] \quad \text{Co-RE}
\]

2 7. Looking back at Question 6, which of these are candidates for using Rice’s Theorem to show their unsolvability? Check all for which Rice Theorem might apply.

a) ___ $X$ b) ___ $X$ c) ____ d) ___ $X$
3 8. We wish to prove that, if \( S \) recursively enumerable and \( \text{dom}(g_S) = S \), that we can create an algorithm, \( f_S \), such that \( x \in S \) iff \( f_S(x) = x \) and \( x \notin S \) iff \( f_S(x) \uparrow \).

\[
f_S(x) = x * \exists t \ STP(g_S, x, t)
\]

6 9. Let set \( A \) be an re non-recursive (undecidable) set that does not contain the value 0, and let \( B \) be a non-empty recursive (decidable) set.
Consider \( C = \{ z \mid z = x - y \), where \( x \in A \) and \( y \in B \} \). // -- is limited subtraction (produces max (0, x – y), which is primitive recursive)
For (a)-(c), either show sets \( A \) and \( B \) and the resulting set \( C \), such that \( C \) has the specified property (argue convincingly that it has the correct property) or demonstrate (prove by construction) that this property cannot hold.

a. Can \( C \) be recursive? Circle \( Y \) or \( N \).
   
   Let \( A = K \) and \( B = \aleph \)
   \( C = K - \aleph = \aleph \) which is recursive

b. Can \( C \) be re non-recursive? Circle \( Y \) or \( N \).
   
   Let \( A = K \) and \( B = \{0\} \)
   \( C = K - \{0\} \cong m K \) which is re, non-recursive.

c. Can \( C \) be non-re? Circle \( Y \) or \( \nabla \).
   You may assume \( A = \text{range}(f_A) \), \( B = \text{range}(f_B) \), for some algorithms \( f_A, f_B \).

\( C \) can be enumerated by \( f_C(<x,y>) = f_A(x) - f_B(y) \) where the minus is limited subtraction.
Thus, \( C \) is always re.
10. Define **NonTrivialDomain (NTD)** = \{ f \mid |\text{Domain}(f)| > 1 \}.

2 a.) Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of **NTD**.

\[ \exists <x,y,t> [x \neq y \& \text{STP}(f,x,t) \& \text{STP}(f,y,t)] \]

4 b.) Use Rice’s Theorem to prove that **NTD** is undecidable.

\[ \text{Dom}(S) = \mathcal{N} \text{ and } |\mathcal{N}| > 1 \text{ so } S \in \text{NTD} \text{ (Note all PRFs are in NTD)} \]

\[ \text{Dom}(\uparrow) = \{ \} \text{ and } |\{ \}| = 0 \text{ so } \uparrow \notin \text{NTD} \]

Thus, **NTD** is non-trivial

Let \( f \) and \( g \) be indices of functions where \( \text{dom}(f) = \text{dom}(g) \).

\[ f \in \text{NTD} \iff |\text{dom}(f)| > 1 \quad \text{By definition} \]

\[ \iff |\text{dom}(g)| > 1 \quad \text{As } \text{dom}(f) = \text{dom}(g) \]

\[ \iff g \in \text{NTD} \]

This shows **NTD** is undecidable due to Rice’s Theorem -- Weak Form based on domains

3 c.) Consider **NonTrivialRange (NTR)** = \{ f \mid |\text{Range}(f)| > 1 \}. Show that **NTR** \( \leq_m \) **NTD**.

Let \( f \) be an arbitrary index. Define \( \forall x \ Gf(x) = \exists y, t \ [ \text{STP}(f,x,t) \& \text{VALUE}(f,y,t) = x ] \)

Note this is \( \exists y \ [ f(y) = x ] \) but there is an issue in writing it this way with \( f \) a procedure

\[ f \in \text{NTR} \iff |\text{Range}(f)| > 1 \iff \exists <x_0, x_1>, \ x_0 \neq x_1, \text{ where } x_0 \& x_1 \in \text{Range}(f) \]

\[ \Rightarrow Gf(x_0) \downarrow \text{ and } Gf(x_1) \downarrow, \ x_0 \neq x_1 \Rightarrow Gf \in \text{NTD. Thus, } f \in \text{NTR} \Rightarrow Gf \in \text{NTD} \]

\[ f \notin \text{NTR} \iff |\text{Range}(f)| \leq 1. \text{ But then either } \forall y f(y) \uparrow \Rightarrow |\text{dom}(Gf)| = 0 \]

Or \( \exists x \forall y f(y) \downarrow \Rightarrow f(y) = x \) and so \( Gf(x) \downarrow \) but \( Gf(z) \uparrow \) if \( z \neq x \Rightarrow |\text{dom}(Gf)| = 1 \)

Thus, \( f \notin \text{NTR} \Rightarrow Gf \notin \text{NTD and so } f \in \text{NTR} \iff Gf \in \text{NTD} \)

3 d.) Show that **NTD** \( \leq_m \) **NTR**.

Let \( f \) be an arbitrary index, Define \( \forall x \ Gf(x) = x \ast \exists t \ [ \text{STP}(f,x,t) ] = f(x) - f(x) + x \)

Clearly \( Gf(x) = x \iff x \in \text{dom}(f) \) and \( Gf(x) \uparrow \) if \( x \notin \text{dom}(f) \)

Thus, \( \text{Range}(Gf) = \text{dom}(Gf) = \text{dom}(f) \) and so their cardinalities are the same.

And so, \( f \in \text{NTD} \iff Gf \in \text{NTR} \)