COT 6410 Spring 2022 **Midterm** **Name:____KEY_________**

Raw Score ___ / 60 **Grade:_________**

6 **1.** In each case below, consider **R1** to be Regular, **R2** to be finite, and **L1** and **L2** to be non-regular CFLs. Fill in the three columns with **Y** or **N**, indicating what kind of language **L** can be. No proofs are required. Read $\subseteq$ as "contained in and may equal."
Put **Y** in all that are possible and **N** in all that are not.

Definition of L	Regular?	CFL, non-Regular?	Not even a CFL?
$L = L1 \cap L2$	*Y*	*Y*	*Y*
$L = L1 - R2$	*N*	*Y*	*N*
$L = \Sigma^* - R1$	*Y*	*N*	*N*
$L \subseteq R1$	*Y*	*Y*	*Y*

3 **2.** Choosing from among **(D) decidable**, **(U) undecidable**, categorize each of the following decision problems. No proofs are required. **L** is a language over **Σ**.

Problem / Language Class	Regular	Context Free	Context Sensitive	Phrase Structured
L contains Σ ?	*D*	*D*	*D*	*U*
|L| is infinite ?	*D*	*D*	*U*	*U*

4 **3.** Prove that any class of languages, ***C***, closed under union, concatenation, intersection with regular languages, homomorphism and substitution (e.g., the Context-Free Languages) is closed under **Cut with Regular Sets**, denoted by the operator |||, where $L \in C$, **R** is Regular, **L** and **R** are both over the alphabet **Σ**, and
$L|||R = \{ x_1 x_2 \ldots x_k \mid k \geq 1, \forall i\ y_i \in R \text{ and } x_1 y_1 x_2 y_2 \ldots x_k y_k \in L \text{ where each } x_i \in \Sigma^+ \}$.
You may assume substitution $f(a) = \{a, a'\}$, and homomorphisms $g(a) = a'$ and $h(a) = a$, $h(a') = \lambda$. Here $a \in \Sigma$ and **a'** is a new character associated with each such $a \in \Sigma$.
You only need give me the definition of **L|||R** in an expression that obeys the above closure properties; you do not need to prove or even justify your expression.

L|||R = $h(f(L) \cap (\Sigma^+ g(R))^+)$

4 **4.** Specify True (**T**) or False (**F**) for each statement.

Statement	T or F
The Context Sensitive Languages are closed under union	*T*
The Post Correspondence Problem is undecidable if |Σ| = 2	*T*
The function $Univ(f,x) = \varphi_f(x)$ is primitive recursive	*F*
If **Halt** $\leq_m$ **P** then **P** must be RE	*F*
The **RE** sets are closed under complement	*F*
Myhill-Nerode proves that every regular language has a minimum state DFA	*T*
If **P** is solvable then Rice's Theorem cannot apply to P	*T*
The incorrect traces of a Turing Machine's Computations form a CFL	*T*

4 **5.** Let $\mathbf{P} = \langle\langle x_1,x_2,\ldots,x_n\rangle, \langle y_1,y_2,\ldots,y_n\rangle\rangle$, $x_i,y_1 \in \Sigma^+$, $1 \le i \le n$, be an arbitrary instance of **PCP**. We can use **PCP's undecidability** to show the undecidability of the problem to determine if the language associated with a **Context Sensitive Grammar** is non-empty. Present a grammar, **G**, associated with an arbitrary instance of PCP, **P**, such that $\mathcal{L}(\mathbf{G})$ is non-empty if and only if there is a solution to **P**. Define $\mathbf{G} = (\{S, T\} \cup \Sigma, \{*\}, R, S)$ where **R is** the set of rules (this is your job):

S	$\rightarrow$	$x_i\, S\, y_i^R \mid x_i\, T\, y_i^R$	$1 \le i \le n$
$a\, T\, a$	$\rightarrow$	$*\, T\, *$	$\forall a \in \Sigma$
$a\, *$	$\rightarrow$	$*\, \alpha$	
$*\, a$	$\rightarrow$	$\alpha\, *$	
T	$\rightarrow$	$*$	

12 **6.** Choosing from among **(REC) recursive**, **(RE) re non-recursive, (coRE) co-re non-recursive**, **(NRNC) non-re/non-co-re**, categorize each of the sets in a) through d). Justify each answer by showing some minimal quantification of some known recursive predicate.

a.) D = { <f, x> | $\varphi_f(x)$ converges in at most 10 steps }

$STP(f, x, 10)$ ______ **REC**

b.) B = { f | φ_f is not total }

$\exists x\ \forall t\ \sim STP(f, x, t)$ ______ **NRNC**

c.) C = { f | domain(φ_f) is non-empty }

$\exists \langle x, t\rangle\ STP(f, x, t)$ ______ **RE**

d.) A = { f | |range(φ_f)| ≤ 1 }

$\forall\langle x,y,t\rangle[STP(f,x,t) \,\&\, STP(f,y,t) \Rightarrow VALUE(f,x,t)=VALUE(f,y,t)]$ ______ **CO-RE**

2 **7.** Looking back at Question 6, which of these are candidates for using Rice's Theorem to show their unsolvability? Check all for which Rice Theorem might apply.

a) ____ **b)** X **c)** X **d)** X

8. We wish to prove that, if **S** and its complement **S'** are both be non-empty and recursively enumerable, then S is recursive (decidable). There are two approaches. The first is based on the fact that there are algorithms, $\mathbf{f_S}$ and $\mathbf{f_{S'}}$, that enumerate **S** and **S'**, respectively. The second is based on the fact that there are procedures (partial recursive functions), $\mathbf{g_S}$ and $\mathbf{g_{S'}}$, whose domains are **S** and **S'**, respectively.

3 **a.)** Define a characteristic function for **S** based on the existence of $\mathbf{f_S}$ and $\mathbf{f_{S'}}$.

$\chi_S(x)$ = $f_S\,(\,\mu\, y\; [\, f_S(y) = x \;||\; f_{S'}(y) = x\,]\,) = x$

3 **b.)** Define a characteristic function for **S** based on the existence of $\mathbf{g_S}$ and $\mathbf{g_{S'}}$.

$\chi_S(x)$ = $STP(\, g_S,\, x,\, \mu\, t\; [\, STP(\, g_S,\, x\,,\, t\,) \;||\; STP(g_{S'},\, x,\, t)\,]\,)$

6 **9.** Let sets **A** be a **non-empty** recursive (decidable) set and let **B** be re non-recursive (undecidable). Consider **C** = { **z** | **z** = **min(x,y),** where **x** ∈ **A** and **y** ∈ **B** }.
For (a)-(c), either show sets **A** and **B** and the resulting set **C**, such that **C** has the specified property (argue convincingly that it has the correct property) or demonstrate (prove by construction) that this property cannot hold.

a. Can **C** be recursive? Circle ***Y*** or **N.**

A = { 0 }   B = K

C = A = { 0 } which is recursive

b. Can **C** be re non-recursive? Circle ***Y*** or **N.**

A = { 2x | x ∈ $\aleph$ }   B = { 2 x + 1 | x ∈ K }

C = A ∪ B as they are disjoint sets and each element in each set is the minimum of some pair with the other set. The membership problem for K reduces to that of C and vice versa.
Thus, C is re, non-recursive

c. Can **C** be non-re? Circle **Y** or ***N*.**
You may assume **A** = **range(f_A),** **B** = **range(f_B),** for some algorithms $\mathbf{f_A}$, $\mathbf{f_B}$.

$f_{min(A,B)}\,(\,\langle x, y\rangle\,) = min\,(\,f_A\,(x),\; f_B\,(y)\,)$

The range of $f_{min(A,B)}$ is then the set of minimums of the sets A and B.
This shows C is the range of some procedure and so is RE

10. Define **PseudoFIB (PF) = { f |** for some input **x**, **f(x+2) = f(x+1)+f(x) }**.

2 **a.)** Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at **c.s** and **d.)** to get a clue as to what this must be.)

$\exists$<x, t> [STP(f, x+2, t) & STP(f, x+1, t) & STP(f, x, t) &

VALUE(f, x+2, t) = VALUE(f, x+2, t) + VALUE(f, x, t)]

5 **b.)** Use Rice's Theorem to prove that **PF** is undecidable.

First, PF is non-trivial as C0 $\in$ PF and C1 $\notin$ PF

Next, let f and g, be arbitrary function indices such that $\forall x\, f(x) = g(x)$

$f \in PF \Leftrightarrow \underline{\exists x}\, [\, f(x+2) = f(x)+f(x+1\,])\ \Leftrightarrow$

$\underline{\exists x}\, [\, g(x+2) = gf(x)+g(x+1\,])$ since $\forall x\, f(x) = g(x)$

$\Leftrightarrow g \in PF$

By the Strong Form of Rice's Theorem, PF is thus shown to be undecidable

5 **c.)** Show that **$K \leq_m$ PF**, where **K = { f | f(f)$\downarrow$ }**.

Let f be an arbitrary function index

Define $\underline{\forall x}\, F_f(x) = f(f) - f(f)$

$f \in K \Leftrightarrow \underline{\forall x}\, F_f(x) = 0 \Rightarrow F_f \in PF$

$f \notin K \Leftrightarrow \underline{\forall x}\, F_f(x) \uparrow \Rightarrow F_f \notin PF$

Thus $K \leq_m$ PF as was to be shown

1 **d.)** From **a.)** through **c.)** what can you conclude about the complexity of **PF (Recursive, RE, RE-COMPLETE, CO-RE, CO-RE-COMPLETE, NON-RE/NON-CO-RE)**?

RE-COMPLETE since PF is RE, $K \leq_m$ PF, and K is RE-COMPLETE