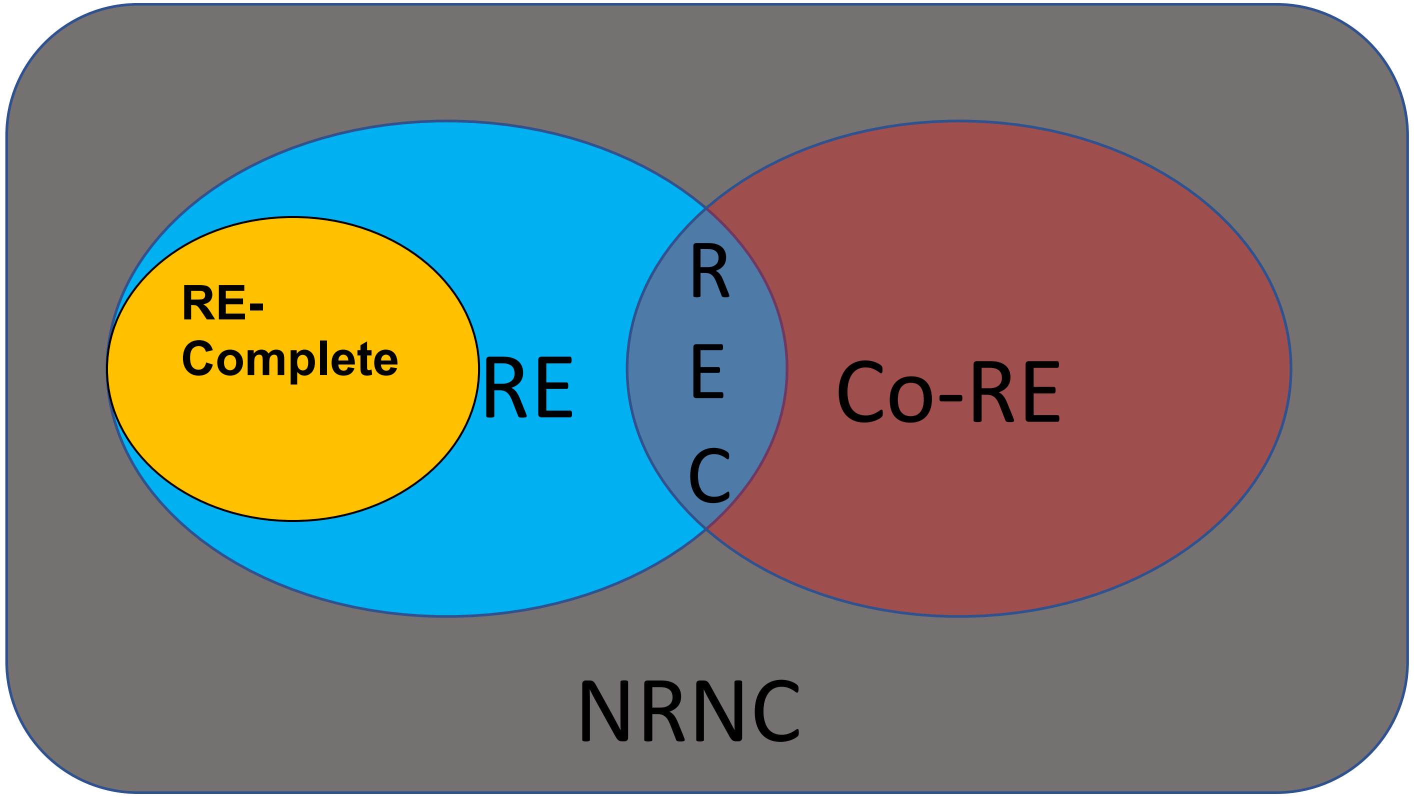
UNIVERSE OF SETS



NR (non-recursive)

= (NRNC U Co-RE) - REC

Some Quantification Examples

- $\langle f,x\rangle \in \text{Halt} \Leftrightarrow \exists t\,[\,STP(f,x,t)\,]$ — RE
- $f \in \text{Total} \Leftrightarrow \forall x \exists t\,[\,STP(f,x,t)\,]$ — NRNC
- $f \in \text{NotTotal} \Leftrightarrow \exists x \forall t\,[\,{\sim}STP(f,x,t)\,]$ — NRNC
- $f \in \text{RangeAll} \Leftrightarrow \forall x \exists \langle y,t\rangle\,[\,STP(f,y,t)\ \&\ VALUE(f,y,t)=x\,]$ — NRNC
- $f \in \text{RangeNotAll} \Leftrightarrow \exists x \forall \langle y,t\rangle\,[STP(f,y,t) \Rightarrow VALUE(f,y,t) \neq x\,]$ — NRNC
- $f \in \text{HasZero} \Leftrightarrow \exists \langle x,t\rangle\,[\,STP(f,x,t)\ \&\ VALUE(f,x,t)=0\,]$ — RE
- $f \in \text{IsZero} \Leftrightarrow \forall x \exists t\,[\,STP(f,x,t)\ \&\ VALUE(f,x,t)=0\,]$ — NRNC
- $f \in \text{Empty} \Leftrightarrow \forall \langle x,t\rangle\,[\,{\sim}STP(f,x,t)\,]$ — Co-RE
- $f \in \text{NotEmpty} \Leftrightarrow \exists\,\langle x,t\rangle\,[\,STP(f,x,t)\,]$ — RE

More Quantification Examples

- $f \in \text{Identity} \Leftrightarrow \forall x \exists t\ [\ STP(f,x,t)\ \&\ VALUE(f,x,t)=x\ ]$ — NRNC
- $f \in \text{NotIdentity} \Leftrightarrow \exists x \forall t\ [\ \sim STP(f,x,t)\ |\ VALUE(f,x,t) \neq x\ ]$ or
  $\exists x \forall t\ [\ STP(f,x,t) \Rightarrow VALUE(f,x,t) \neq x\ ]$ — NRNC
- $f \in \text{Constant} = \forall <x,y> \exists t\ [STP(f,x,t)\ \&\ STP(f,y,t)\ \&$
  $VALUE(f,x,t)=VALUE(f,y,t)]$ — NRNC
- $f \in \text{Infinite} \Leftrightarrow \forall x \exists <y,t>\ [\ y \geq x\ \&\ STP(f,y,t)\ ]$ — NRNC
- $f \in \text{Finite} \Leftrightarrow \exists x \forall <y,t>\ [\ y < x\ |\ \sim STP(f,y,t)\ ]$ or
  $\exists x \forall <y,t>\ [\ STP(f,y,t) \Rightarrow y < x\ ]$ or $[\ y \geq x \Rightarrow \sim STP(f,y,t)\ ]$ — NRNC
- $f \in \text{RangeInfinite} \Leftrightarrow \forall x \exists <y,t>\ [\ STP(f,y,t)\ \&\ VALUE(f,y,t) \geq x\ ]$ — NRNC
- $f \in \text{RangeFinite} \Leftrightarrow \exists x \forall <y,t>\ [\ STP(f,y,t) \Rightarrow VALUE(f,y,t) < x\ ]$ — NRNC
- $f \in \text{Stutter} \Leftrightarrow \exists <x,y,t>\ [\ x \neq y\ \&\ STP(f,x,t)\ \&\ STP(f,y,t)\ \&$
  $VALUE(f,x,t) = VALUE(f,y,t)\ ]$ — RE

Even More Quantification Examples

• <f,x> ∈ Fast20 ⇔ [STP(f,x,20)]	REC
• f ∈ FastOne20 ⇔ ∃x [STP(f,x,20)]	RE
• f ∈ FastAll20 ⇔ ∀x [STP(f,x,20)]	Co-RE
• <f,x,K,C> ∈ LinearKC ⇔ [STP(f,x,K*x+C)]	REC
• <f,K,C>∈ LinearKCOne ⇔ ∃x [STP(f,x,K*x+C)]	RE
• <f,K,C> ∈ LinearKCAll ⇔ ∀x [STP(f,x,K*x+C)]	Co-RE

- None of the above can be shown undecidable using Rice's Theorem
- In fact, reduction from known undecidables is also a problem for all but the first one which happens to be decidable.

Some Reductions and Rice Example

- **NotEmpty ≤ Halt**
  Let f be an arbitrary index
  Define $\forall y\ g_f(y) = \exists \langle x,t \rangle\ STP(f,x,t)$
  $f \in$ Notmpty $\Leftrightarrow \langle g_f,0 \rangle \in$ Halt
- **Halt ≤ NotEmpty**
  Let f,x be an arbitrary index and input value
  Define $\forall y\ g_{f,x}(y) = f(x)$
  $\langle f,x \rangle \in$ Halt $\Leftrightarrow g_{f,x} \in$ NotEmpty
- **Note: NotEmpty is RE-Complete**
- **Rice: NotEmpty is non-trivial Zero ∈ NotEmpty; $\uparrow \notin$ NotEmpty**
  Let f,g be arbitrary indices such that Dom(f)=Dom(g)
  $f \in$NotEmpty $\Leftrightarrow$ Dom(f) ≠ ∅ By Definition
  $\Leftrightarrow$ Dom(g) ≠ ∅ Dom(g)=Dom(f)
  $\Leftrightarrow g \in$ NotEmpty
  Thus, Rice's Theorem states that NotEmpty is undecidable.

More Reductions and Rice Example

- **Identity ≤ Total**
  Let f be an arbitrary index
  Define $g_f(x) = \mu y\ [\ f(x) = x\ ]$
  $f \in$ Identity $\Leftrightarrow g_f \in$ Total
- **Total ≤ Identity**
  Let f be an arbitrary index
  Define $g_f(x) = f(x)-f(x) + x$
  $f \in$ Total $\Leftrightarrow g_{f,x} \in$ Identity
- **Rice: Identity is non-trivial $I(x)=x \in$ Identity; Zero $\notin$ Identity**
  Let f,g be arbitrary indices such that $\forall x\ f(x) = g(x)$
  $f \in$Identity $\Leftrightarrow$ $\forall x\ f(x)=x$ By Definition
  $\Leftrightarrow$ $\forall x\ g(x)=x$ $\forall x\ g(x) = f(x)$
  $\Leftrightarrow$ g $\in$ Identity
  Thus, Rice's Theorem states that Identity is undecidable

Even More Reductions and Rice Example

- **Stutter ≤ Halt**
  Let f be an arbitrary index
  Define ∀y $g_f(y)$ = ∃<x,y,t> [x≠y & STP(f,x,t) & STP(f,y,t) & VALUE(f,x,t) = VALUE(f,y,t)]
  f ∈ Stutter ⇔ <g_f,0> ∈ Halt
- **Halt ≤ Stutter**
  Let f,x be an arbitrary index and input value
  Define ∀y $g_{f,x}(y)$ = f(x)
  <f,x> ∈ Halt ⇔ $g_{f,x}$ ∈ Stutter
- **Note: Stutter is RE-Complete**
- **Rice: Stutter is non-trivial Zero∈Stutter; I(x)=x ∉ Stutter**
  Let f,g be arbitrary indices such that ∀x f(x) = g(x)
  f ∈Stutter ⇔ ∃<x,y> [x≠y & f(x)=f(y)] — **By Definition**
  ⇔ ∃<x,y> [x≠y & g(x)=g(y)] — **∀x g(x) = f(x)**
  ⇔ g ∈Stutter
  Thus, Rice's Theorem states that Identity is undecidable

Yet More Reductions and Rice Example

- **Constant ≤ Total**
  Let f be an arbitrary index
  Define $g_f(0) = f(0)$
  $g_f(y+1) = \mu z\ [\ f(y+1) = f(y)\ ]$
  $f \in$ Constant $\Leftrightarrow g_f \in$ Total
- **Total ≤ Identity**
  Let f be an arbitrary index
  Define $g_f(x) = f(x)-f(x)$
  $f \in$ Total $\Leftrightarrow g_f \in$ Constant
- **Rice: Constant is non-trivial Zero ∈ Constant; $I(x)=x \notin$ Constant**
  Let f,g be arbitrary indices such that $\forall x\ f(x) = g(x)$
  $f \in$Constant $\Leftrightarrow$ $\exists C \forall x\ f(x)=C$ By Definition
  $\Leftrightarrow$ $\exists C \forall x\ g(x)=C$ $\forall x\ g(x) = f(x)$
  $\Leftrightarrow$ $g \in$ Constant
  Thus, Rice's Theorem states that Identity is undecidable

Last Reductions and Rice Example

- **RangeAll ≤ Total**
  Let f be an arbitrary index
  Define $g_f(x) = \exists y\ [\ f(y) = x\ ]$
  $f \in$ RangeAll $\Leftrightarrow g_f \in$ Total
- **Total ≤ RangeAll**
  Let f be an arbitrary index
  Define $g_f(x) = f(x)-f(x) + x$
  $f \in$ Total $\Leftrightarrow g_f \in$ RangeAll
- **Rice: RangeAll is non-trivial $I(x)=x \in$ RangeAll; Zero $\notin$ RangeAll**
  Let f,g be arbitrary indices such that Range(f) = Range(g)
  $f \in$ RangeAll $\Leftrightarrow$ Range(f) = $\aleph$ By Definition
  $\Leftrightarrow$ Range(f) = $\aleph$ Range(g) = Range(f)
  $\Leftrightarrow g \in$ RangeAll
  Thus, Rice's Theorem states that Identity is undecidable

Challenge

Semi-Constant(SC) = { f | ∃C, ∀x f(x)↓ ⇒ f(x) = C }

Note: **↑** ∈ **SC** and **C_0(x)=0** ∈ **SC**

Can describe as **f ∈ SC** ⇔
∃C ∀<x,t> [STP(f,x,t) ⇒ VALUE(f,x,t) = C]

This implies **SC** is as hard as **Non-TOT={ f |∃x f(x)↑ }** as

f ∈ Non-TOT ⇔ ∃x ∀t [~STP(f,x,t)]

However, **SC** only takes one quantifier and is undecidable (one of the weaker versions of Rice shows its undecidability).

I can tell you that **SC** $\equiv_m$ **HALT** or **SC** $\equiv_m$ **Non-HALT** where **Non-HALT = { <f,x> | f(x)↑ }**.

Your job is to figure out which and rewrite the quantifier expression. You should also apply Rice’s to verify undecidability.