UNIVERSE OF SETS

NR (non-recursive) = (NRNC U Co-RE) - REC

NRNC

RE-Complete

RE

Co-RE

REC
Some Quantification Examples

- \(<f,x> \in \text{Halt} \iff \exists t [ \text{STP}(f,x,t) ]\) RE
- \(f \in \text{Total} \iff \forall x \exists t [ \text{STP}(f,x,t) ]\) NRNC
- \(f \in \text{NotTotal} \iff \exists x \forall t [ \sim \text{STP}(f,x,t) ]\) NRNC
- \(f \in \text{RangeAll} \iff \forall x \exists y,t [ \text{STP}(f,y,t) \land \text{VALUE}(f,y,t) = x ]\) NRNC
- \(f \in \text{RangeNotAll} \iff \exists x \forall y,t [ \text{STP}(f,y,t) \Rightarrow \text{VALUE}(f,y,t) \neq x ]\) NRNC
- \(f \in \text{HasZero} \iff \exists x,t [ \text{STP}(f,x,t) \land \text{VALUE}(f,x,t) = 0 ]\) RE
- \(f \in \text{IsZero} \iff \forall x \exists t [ \text{STP}(f,x,t) \land \text{VALUE}(f,x,t) = 0 ]\) NRNC
- \(f \in \text{Empty} \iff \forall x,t [ \sim \text{STP}(f,x,t) ]\) Co-RE
- \(f \in \text{NotEmpty} \iff \exists x,t [ \text{STP}(f,x,t) ]\) RE
More Quantification Examples

• $f \in \text{Identity} \iff \forall x \exists t \left[ \text{STP}(f,x,t) \ & \ \text{VALUE}(f,x,t) = x \right] \quad \text{NRNC}$

• $f \in \text{NotIdentity} \iff \exists x \forall t \left[ \sim \text{STP}(f,x,t) \ | \ \text{VALUE}(f,x,t) \neq x \right] \quad \text{NRNC}$

• $f \in \text{Constant} = \forall <x,y> \exists t \left[ \text{STP}(f,x,t) \ & \ \text{STP}(f,y,t) \ & \ \text{VALUE}(f,x,t) = \text{VALUE}(f,y,t) \right] \quad \text{NRNC}$

• $f \in \text{Infinite} \iff \forall x \exists <y,t> \left[ y \geq x \ & \ \text{STP}(f,y,t) \right] \quad \text{NRNC}$

• $f \in \text{Finite} \iff \exists x \forall <y,t> \left[ y < x \ | \ \sim \text{STP}(f,y,t) \right] \quad \text{NRNC}$

• $f \in \text{RangeInfinite} \iff \forall x \exists <y,t> \left[ \text{STP}(f,y,t) \ & \ \text{VALUE}(f,y,t) \geq x \right] \quad \text{NRNC}$

• $f \in \text{RangeFinite} \iff \exists x \forall <y,t> \left[ \text{STP}(f,y,t) \Rightarrow \text{VALUE}(f,y,t) < x \right] \quad \text{NRNC}$

• $f \in \text{Stutter} \iff \exists <x,y,t> \left[ x \neq y \ & \ \text{STP}(f,x,t) \ & \ \text{STP}(f,y,t) \ & \ \text{VALUE}(f,x,t) = \text{VALUE}(f,y,t) \right] \quad \text{RE}$
Even More Quantification Examples

- \( <f,x> \in \text{Fast20} \iff [ \text{STP}(f,x,20) ] \)  
  RE
- \( f \in \text{FastOne20} \iff \exists x [ \text{STP}(f,x,20) ] \)  
  RE
- \( f \in \text{FastAll20} \iff \forall x [ \text{STP}(f,x,20) ] \)  
  Co-RE
- \( <f,x,K,C> \in \text{LinearKC} \iff [ \text{STP}(f,x,K*x+C) ] \)  
  REC
- \( <f,K,C> \in \text{LinearKCOone} \iff \exists x [ \text{STP}(f,x,K*x+C) ] \)  
  RE
- \( <f,K,C> \in \text{LinearKCAll} \iff \forall x [ \text{STP}(f,x,K*x+C) ] \)  
  Co-RE

- None of the above can be shown undecidable using Rice’s Theorem
- In fact, reduction from known undecidables is also a problem for all but the first one which happens to be decidable.
Some Reductions and Rice Example

• **NotEmpty ≤ Halt**
  Let $f$ be an arbitrary index
  Define $\forall y \ g_f(y) = \exists <x,t> \ STP(f,x,t)$
  $f \in \text{NotEmpty} \iff <g_f,0> \in \text{Halt}$

• **Halt ≤ NotEmpty**
  Let $f, x$ be an arbitrary index and input value
  Define $\forall y \ g_{f,x}(y) = f(x)$
  $<f,x> \in \text{Halt} \iff g_{f,x} \in \text{NotEmpty}$

• **Note: NotEmpty is RE-Complete**

• **Rice:** NotEmpty is non-trivial $\text{Zero} \in \text{NotEmpty}; \uparrow \notin \text{NotEmpty}$
  Let $f, g$ be arbitrary indices such that $\text{Dom}(f)=\text{Dom}(g)$
  $f \in \text{NotEmpty} \iff \text{Dom}(f) \neq \emptyset$ By Definition
  $\iff \text{Dom}(g) \neq \emptyset$ $\text{Dom}(g)=\text{Dom}(f)$
  $\iff g \in \text{NotEmpty}$
  Thus, Rice's Theorem states that NotEmpty is undecidable.
More Reductions and Rice Example

• **Identity ≤ Total**
  Let f be an arbitrary index
  Define $g_f(x) = \mu y [ f(x) = x ]$
  $f \in \text{Identity} \iff g_f \in \text{Total}$

• **Total ≤ Identity**
  Let f be an arbitrary index
  Define $g_f(x) = f(x) - f(x) + x$
  $f \in \text{Total} \iff g_{f,x} \in \text{Identity}$

• **Rice: Identity is non-trivial**
  $I(x) = x \in \text{Identity}; \ Zero \notin \text{Identity}$
  Let f,g be arbitrary indices such that $\forall x f(x) = g(x)$
  $f \in \text{Identity} \iff \forall x f(x) = x \hspace{1cm} \text{By Definition}$
  $\iff \forall x g(x) = x \hspace{1cm} \forall x g(x) = f(x)$
  $\iff g \in \text{Identity}$
  Thus, Rice’s Theorem states that Identity is undecidable
Even More Reductions and Rice Example

- Stutter ≤ Halt
  Let $f$ be an arbitrary index
  Define $\forall y \ g_f(y) = \exists <x,y,t> \ [ x \neq y \ & \ STP(f,x,t) \ & \ STP(f,y,t) \ & \ VALUE(f,x,t) = VALUE(f,y,t) ]$
  
  $f \in \text{Stutter} \iff <g_f,0> \in \text{Halt}$

- Halt ≤ Stutter
  Let $f,x$ be an arbitrary index and input value
  Define $\forall y \ g_{f,x}(y) = f(x)$
  
  $<f,x> \in \text{Halt} \iff g_{f,x} \in \text{Stutter}$

- Note: Stutter is RE-Complete

- Rice: Stutter is non-trivial
  Zero $\notin \text{Stutter}$; $I(x)=x \notin \text{Stutter}$
  Let $f,g$ be arbitrary indices such that $\forall x \ f(x) = g(x)$
  
  $f \in \text{Stutter} \iff \exists <x,y> \ [ x \neq y \ & \ f(x)=f(y) ] \iff \exists <x,y> \ [ x \neq y \ & \ g(x)=g(y) ]$  
  By Definition $\forall x \ g(x) = f(x)$
  
  $\iff g \in \text{Stutter}$

Thus, Rice’s Theorem states that Identity is undecidable
Yet More Reductions and Rice Example

• **Constant ≤ Total**
  Let $f$ be an arbitrary index
  Define $g_f(0) = f(0)$
  
  $$g_f(y+1) = \mu z \ [ f(y+1) = f(y) ]$$
  $f \in \text{Constant} \iff g_f \in \text{Total}$

• **Total ≤ Identity**
  Let $f$ be an arbitrary index
  Define $g_f(x) = f(x) - f(x)$
  $f \in \text{Total} \iff g_f \in \text{Constant}$

• **Rice: Constant is non-trivial**
  $\text{Zero} \in \text{Constant}; \ I(x) = x \notin \text{Constant}$
  Let $f, g$ be arbitrary indices such that $\forall x \ f(x) = g(x)$
  
  $$f \in \text{Constant} \iff \exists C \forall x \ f(x) = C$$
  By Definition
  
  $$\iff \exists C \forall x \ g(x) = C \quad \forall x \ g(x) = f(x)$$
  
  $$\iff g \in \text{Constant}$$
  
  Thus, Rice’s Theorem states that Identity is undecidable
RangeAll ≤ Total
Let f be an arbitrary index
Define \( g_f(x) = \exists y \ [ f(y) = x ] \)
\( f \in \text{RangeAll} \iff g_f \in \text{Total} \)

Total ≤ RangeAll
Let f be an arbitrary index
Define \( g_f(x) = f(x) - f(x) + x \)
\( f \in \text{Total} \iff g_f \in \text{RangeAll} \)

Rice: RangeAll is non-trivial \( I(x) = x \in \text{RangeAll}; \ Zero \notin \text{RangeAll} \)
Let \( f, g \) be arbitrary indices such that \( \text{Range}(f) = \text{Range}(g) \)
\( f \in \text{RangeAll} \iff \text{Range}(f) = \mathbb{N} \) By Definition
\( \iff \text{Range}(f) = \mathbb{N} \) Range(g) = Range(f)
\( \iff g \in \text{RangeAll} \)
Thus, Rice’s Theorem states that Identity is undecidable
Challenge

Semi-Constant (SC) = \{ f | \exists C, \forall x \ f(x) \downarrow \Rightarrow f(x) = C \}

Note: \uparrow \in SC and C_0(x) = 0 \in SC

Can describe as f \in SC \iff
\exists C \forall <x,t> [ STP(f,x,t) \Rightarrow VALUE(f,x,t) = C ]

This implies SC is as hard as Non-TOT = \{ f | \exists x \ f(x) \uparrow \} as
f \in Non-TOT \iff \exists x \forall t [ \neg STP(f,x,t) ]

However, SC only takes one quantifier and is undecidable (one of the weaker versions of Rice shows its undecidability).

I can tell you that SC \equiv_m HALT or SC \equiv_m Non-HALT where Non-HALT = \{ <f,x> | f(x) \uparrow \}.

Your job is to figure out which and rewrite the quantifier expression. You should also apply Rice’s to verify undecidability.