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# A REDUCTION CLASS CONTAINING FORMULAS WITH ONE MONADIC PREDICATE AND ONE BINARY FUNCTION SYMBOL

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**Abstract.** A new reduction class is presented for the satisfiability problem for well-formed formulas of the first-order predicate calculus. The members of this class are closed prenex formulas of the form $\forall x \forall y C$. The matrix $C$ is in conjunctive normal form and has no disjuncts with more than three literals, in fact all but one conjunct is unary. Furthermore $C$ contains but one predicate symbol, that being unary, and one function symbol which symbol is binary.

**Introduction.** An effective method is presented for constructing, from an arbitrary diadic partial implicational propositional calculus $D$, and an arbitrary one variable wff $W$, a first-order formula $F$ such that $W$ is derivable in $D$ if and only if $F$ is a tautology of the first-order predicate calculus. Each such $F$ is a member of the class $\mathscr{D}$ of closed prenex formulas of the form $\exists x \exists y C$ where $C$ is in disjunctive normal form, has all unary conjuncts, except one which contains three literals, and has just a single monadic predicate symbol and a single binary function symbol. Using a result in [4] we are then able to conclude that there exists an effective method which produces, from an arbitrary first-order formula $F_1$, a first-order formula of the above type $F_2$ such that $F_1$ is a tautology iff $F_2$ is also one. Next, since a given formula $F_1$ is satisfiable iff $\neg F_1$ is not a tautology, we get that the class of formulas $\mathscr{C}$, comprised of negations of all formulas from the class $\mathscr{D}$ above, forms a reduction class with respect to satisfiability. This class $\mathscr{C}$ is just the set of all closed prenex formulas $\forall x \forall y C$ where $C$ is in conjunctive normal form with one ternary disjunct, the rest being unary, and $C$ has one monadic predicate and one binary function symbol.

**Partial calculi.** Let $p_1, p_2, \cdots$ be the set of all propositional variables in some formulation of the propositional calculus. $P_n$ will be used to denote the class of all well-formed formulas (wffs) of the implicational propositional calculus which involve only variables among $\{p_1, p_2, \cdots, p_n\}$. A *diadic partial implicational propositional calculus* (PIPC) $I$ is an inference system defined by a finite set of tautologies from $P_2$. These tautologies are called the *axioms* of $I$. The rules of inference of $I$ are modus ponens and substitution of wffs from $P_2$, that is the normal propositional rules of inference restricted to the use of wffs from $P_2$. Let $W$ be an arbitrary member of $P_2$. Then $W$ is said to be *derivable* in $I$ iff $W$ may be deduced from the axioms of $I$ by its rules of inference. The classes of wffs which

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may be derived by diadic PIPC's were the subject of study in [4] and the results presented there form a basis for those shown here.

**Reduction classes.** A class $R$ of formulas of the first-order predicate calculus is called a *reduction class with respect to satisfiability* (respectively *deducibility*) if there exists an effective procedure $\phi$ which maps first-order formulas $F$ into members $\phi(F)$ of $R$ such that $F$ is satisfiable, i.e. true under some interpretation (respectively a tautology) iff $\phi(F)$ is also satisfiable (respectively a tautology).

Reduction classes have been studied by a rather large number of authors including [1], [2] and [5]. The reduction class $\mathscr{C}$ presented here is most closely related to two classes recently announced by Börger [1]. There, it was claimed that the classes $\mathscr{C}_1$ and $\mathscr{C}_2$ of closed prenex formulas of the forms $\forall x \forall y C$ and $\forall x \forall y \forall z D$, respectively, are each reduction classes for satisfiability, where $C$ and $D$ are both in conjunctive normal form with binary disjuncts, $C$ has one monadic predicate, one monadic and one binary function, and finally $D$ has one monadic predicate and one binary function symbol. Our result is both a strengthening and a weakening of Börger's. It is stronger in that the formulas of $\mathscr{C}$ have the binary prefixes of $\mathscr{C}_1$ while only having a single binary function symbol as those in $\mathscr{C}_2$. Our result is weaker since members of $\mathscr{C}$ have matrices which contain a ternary disjunct whereas those in $\mathscr{C}_1$ and $\mathscr{C}_2$ have only binary disjuncts.

**Demonstration that $\mathscr{C}$ is a reduction class.** Let $D$ be an arbitrary diadic PIPC with axioms $A_1, A_2, \cdots, A_n$. Further let the formulas of $D$ be written in prefix (polish) notation.

We define, for each wff $W$ contained in $P_2$, a first-order formula $F$ such that $W$ is derivable in $D$ iff $F$ is a tautology. The formula $F$ to be constructed will contain variables $x$ and $y$, the one monadic predicate symbol $T$ and the one binary function symbol $f$. As our first step we define a function $K$ from the class of formulas $P_2$ into the class of first-order formulas.

$K(p_1) = x$,
$K(p_2) = y$, and
$K(\supset W_1 W_2) = f(K(W_1), K(W_2))$, for $W_1$ and $W_2$ members of $P_2$.

Then, for example, if $W = \supset \supset p_1 p_2 \supset p_1 p_2$, we would have $K(W) = f(f(x, y), f(x, y))$. The *first-order image* of $W$, denoted $W^*$, is defined to be $\forall x \forall y T(K(W))$, $\forall x$ or $\forall y$ being omitted if the variable $x$ or $y$ does not occur in $K(W)$.

Let, as noted above, $A_1, A_2, \cdots, A_n$ be the axioms of $D$. The *first-order associate* of $D$, denoted $\bar{D}$, is defined to be

$$A_1^* \ \& \ A_2^* \ \& \ \cdots \ \& \ A_n^* \ \& \ \forall x \forall y ((T(x) \ \& \ T(f(x, y))) \supset T(y)).$$

We shall now show the following.

LEMMA 1. *Let $W$ be an arbitrary element in $P_2$. Then $W$ is derivable in $D$ iff $\bar{D} \supset W^*$ is a tautology of the first-order predicate calculus.*

PROOF. (I) Assume $W$ is derivable in $D$. Then there exist wffs $W_1, \cdots, W_k$ where $W = W_k$ and each $W_i$ is either

(a) an axiom, or

(b) the result of substituting a wff $Y$ for all occurrences of one of the variables $p_1$ or $p_2$ in some $W_s$, $1 \leq s < i$, or

(c) the result of applying modus ponens to some pair $W_s$, $W_r$, where $s$ and $r$ are less than $i$.

We prove this part of our theorem by induction on $k$.

$k = 1$. Then $W$ must be an axiom. That is, $W$ is $A_t$ for some $t$, $1 \leq t \leq n$. But then we have

1. $\bar{D}$—Hypothesis.
2. $A_t^*$—From the fact that $(\boldsymbol{A} \mathbin{\&} \boldsymbol{B} \mathbin{\&} \boldsymbol{C}) \supset \boldsymbol{B}$ or $(\boldsymbol{A} \mathbin{\&} \boldsymbol{C}) \supset \boldsymbol{A}$ and (1).

Hence $\bar{D} \supset W^*$ is a tautology.

$k > 1$. Assume $\bar{D} \supset W_i^*$ for all $i$, $1 \leq i < k$.

*Part* a. If $W_k$ is an axiom then $\bar{D} \supset W^*$ by our arguments for the $k = 1$ case.

*Part* b. Substitution of a wff $Y$ in $P_2$ for a variable. By hypothesis, $\bar{D} \supset W_s^*$. We need to show that $\bar{D} \supset W^*$ where $W$ is obtained from $W_s$ by substituting $Y$ for all occurrences of $p_1$ (or $p_2$).

1. $\bar{D} \supset W_s^*$—Tautology by inductive hypothesis.
2. $\bar{D}$—Hypothesis.
3. $W_s^*$—*MP* (1 and 2).
4. $T(K(W_s))$—Universal instantiation (3).

5a. $\forall x T(K(W_s))$—Universal generalization (4); or

5b. $\forall y T(K(W_s))$.

6. $T(K(W))$—Universal instantiation. Substitution of $K(Y)$ for $x$ in 5a or $y$ in 5b.
7. $W^*$—Universal generalization (6).

Hence $\bar{D} \supset W^*$ is a tautology.

*Part* c. Modus ponens. We need show that if $\bar{D} \supset W_s^*$ and $\bar{D} \supset W_r^*$ are tautologies and $W_s$ is $\supset W_r W$ then $\bar{D} \supset W^*$ is a tautology.

1. $\bar{D} \supset W_s^*$—Tautology by inductive hypothesis.
2. $\bar{D} \supset W_r^*$—Tautology by inductive hypothesis.
3. $\bar{D}$—Hypothesis.
4. $\forall x \forall y((T(x) \mathbin{\&} T(f(x, y))) \supset T(y))$—From the fact that $(\boldsymbol{A} \mathbin{\&} \boldsymbol{C}) \supset \boldsymbol{C}$ and 3.
5. $W_s^*$—*MP* (1 and 3).
6. $W_r^*$—*MP* (2 and 3).
7. $T(f(K(W_r), K(W)))$—Universal instantiation (5).
8. $T(K(W_r))$—Universal instantiation (6).
9. $T(K(W_r)) \mathbin{\&} T(f(K(W_r), K(W)))$—From fact that $\boldsymbol{A} \supset (\boldsymbol{B} \supset (\boldsymbol{A} \mathbin{\&} \boldsymbol{B}))$ and 7 and 8.
10. $(T(K(W_r)) \mathbin{\&} T(f(K(W_r), K(W)))) \supset T(K(W))$—Universal instantiation (4).
11. $T(K(W))$—*MP* (9 and 10).
12. $W^*$—Universal generalization (11).

And hence $\bar{D} \supset W^*$ is a tautology.

This completes our inductive proof that if $W$ is derivable in $D$ then $\bar{D} \supset W^*$ is a tautology.

(II) It now remains to be shown that if $\bar{D} \supset W^*$ is a tautology then $W$ is derivable in $D$. Our proof in this direction will be semantic rather than syntactic as it was in (I). That is, we shall use the fact that $\bar{D} \supset W^*$ is a tautology iff it is true under all interpretations and in particular under the following interpretation.

*Domain of interpretation.* The set of wffs (in prefix notation) in $P_2$.

*Interpretation of f.* $f$ is interpreted to be the binary operator which maps $x_1$, $x_2$ to $\supset x_1 x_2$, where of course $x_1$ and $x_2$ are variables ranging over wffs in $P_2$.

*Interpretation of T.* $T$ is true for the argument $x$ if and only if $x$ is derivable in $D$.

Now, with the above interpretation in mind, we shall show that $\bar{D} \supset W^*$ is true implies $W$ is derivable in $D$. Our method of attack will be to first show that $\bar{D}$ is true under our chosen interpretation, which we denote $I_D$.

$\bar{D}$ is true iff $A_i^*$ is true, for $1 \leq i \leq n$, and $\forall x \forall y((T(x) \& T(f(x, y))) \supset T(y))$ is true. But, treating the $A_i^*$'s, we have that $A_i^*$ is true under $I_D$ iff all substitution instances of $A_i$ are derivable in $D$. This clearly shows that the $A_i^*$'s are true under $I_D$. Now $\forall x \forall y((T(x) \& T(f(x, y))) \supset T(y))$ is true under $I_D$ if the derivability of wffs $W_1$ and $\supset W_1 W_2$ implies the derivability of $W_2$. But this is just modus ponens.

Now, using the fact that $\bar{D}$ is true under $I_D$, we may complete our proof by showing that the truth of $W^*$ under $I_D$ implies the derivability of $W$ in $D$. But $W^*$ is true implies all substitution instances of $W$ are derivable in $D$ which clearly implies that $W$ is derivable.

This then completes the proof of this theorem.

LEMMA 2. *Let $D$ be an arbitrary diadic PIPC and let $W$ be a wff in $P_1$, i.e. containing only the variable $p_1$. Then some substitution instance of $W$ is derivable in $D$ iff $\bar{D} \supset \exists x T(K(W))$ is a tautology.*

PROOF. First assume that some substitution instance $W'$ of $W$ is derivable in $D$. Then, by Lemma 1, $\bar{D} \supset \forall x \forall y T(K(W'))$ is a tautology, where of course one of $\forall x$ or $\forall y$ might not appear. But then $\bar{D} \supset T(K(W'))$ and consequently $\bar{D} \supset \exists x T(K(W))$ are each tautologies. (The latter is a tautology since $T(K(W'))$ may be rewritten as $T(K(W'))$ by an appropriate substitution for the variable $x$.

Next assume that $\bar{D} \supset \exists x T(K(W))$ is a tautology. Then, using the interpretation appearing in the proof of Lemma 1, we may achieve the result via arguments similar to those given there.

LEMMA 3. *There exist effective procedures $\phi_1$ and $\phi_2$, where $\phi_1$ maps recursively enumerable (r.e.) sets to diadic PIPC's and $\phi_2$ maps natural numbers to member of $P_1$ such that, if $S$ is an arbitrary r.e. set and $x$ is an arbitrary natural number, then $x \in S$ iff $\phi_2(x)$ is derivable in $\phi_1(S)$. Furthermore, $\phi_2(x)$ is derivable iff it has some substitution instance which is derivable.*

PROOF. Let $S$ and $x$ be as in the statement of the lemma. Davis [3] and others have demonstrated procedures $\phi_3$ and $\phi_4$ such that $\phi_3(S)$ is a Turing machine and $\phi_4(x)$ is a configuration of $\phi_3(S)$ which is mortal iff $x \in S$. In [4] we presented procedures $\phi_5$ and $\phi_6$ such that, if $M$ is an arbitrary Turing machine and $C$ is an arbitrary configuration, then $\phi_5(M)$ is a diadic PIPC, $\phi_6(C)$ is a wff containing only the variable $p_1$ and $C$ is mortal in $M$ iff $\phi_6(C)$ is derivable in $\phi_5(M)$. Moreover, if $W$ is some substitution instance of $\phi_6(C)$ then $\phi_6(C)$ is derivable in $\phi_5(M)$ iff $W$ is also. This latter statement may be verified by an examination of Lemmas 1, 3 and 4 of [4] in which the reader should observe the independence of the members of each of the 5 forms discussed there. The proof is then completed by letting $\phi_1(S) = \phi_5(\phi_3(S))$ and $\phi_2(x) = \phi_6(\phi_4(x))$.

LEMMA 4. *There exists a fixed diadic PIPC $\mathscr{P}$ and an effective method $\phi$ such that, if $\phi$ is applied to an arbitrary first-order formula F, then F is a tautology iff some substitution instance of $\phi(F)$ is derivable in $\mathscr{P}$, where $\phi(F)$ is in $P_1$.*

PROOF. Let $\mathscr{F}$ be the class of numbers, under some Gödel numbering $g$, of all tautologies of the first-order predicate calculus. $\mathscr{F}$ is r.e. since it may be defined by a finite set of axioms and a finite set of recursive rules of inference.

Let $F$ be an arbitrary first-order formula and let $g(F)$ be the Gödel number of $F$. Then $F$ is a tautology iff some substitution instance of $\phi_2(g(F))$ is derivable in $\phi_1(\mathscr{F})$, where $\phi_1$ and $\phi_2$ are defined as in Lemma 3. The lemma is then shown by letting $\mathscr{P}$ be $\phi_1(\mathscr{F})$ and $\phi(F)$ be $\phi_2(g(F))$.

THEOREM 1. *The class $\mathscr{D}$ of first-order formulas is a reduction class with respect to deducibility.*

PROOF. Let $F$ be an arbitrary formula and let $\mathscr{P}$ and $\phi$ be as in Lemma 4. By Lemma 4, $F$ is a tautology iff some substitution instance of $\phi(F)$ is derivable in $\mathscr{P}$. By Lemma 2, this is so iff $\bar{\mathscr{P}} \supset \exists x T(K(\phi(F)))$ is a tautology. But $\bar{\mathscr{P}} \supset \exists x T(K(\phi(F)))$ may be seen to be in $\mathscr{D}$ as follows: First it is of the following form:

$$\forall x \forall y L_1 \mathbin{\&} \forall x \forall y L_2 \mathbin{\&} \cdots \mathbin{\&} \forall x \forall y L_n \mathbin{\&} \forall x \forall y ((L_{n+1} \mathbin{\&} L_{n+2}) \supset L_{n+3}) \supset \exists x L_{n+4},$$

where each $L_i$ is a literal. This form may be successively rewritten as

1. $\forall x \forall y (L_1 \mathbin{\&} L_2 \mathbin{\&} \cdots \mathbin{\&} L_n \mathbin{\&} ((L_{n+1} \mathbin{\&} L_{n+2}) \supset L_{n+3})) \supset \exists x L_{n+4}$,
2. $\neg \forall x \forall y (L_1 \mathbin{\&} L_2 \mathbin{\&} \cdots \mathbin{\&} L_n \mathbin{\&} \neg(L_{n+1} \mathbin{\&} L_{n+2} \mathbin{\&} \neg L_{n+3})) \vee \exists x L_{n+4}$, and then
3. $\exists x \exists y (\neg L_1 \vee \neg L_2 \vee \cdots \vee \neg L_n \vee (L_{n+1} \mathbin{\&} L_{n+2} \mathbin{\&} \neg L_{n+3}) \vee L_{n+4})$.

Clearly form 3 belongs to class $\mathscr{D}$ proving the theorem.

THEOREM 2. *The class $\mathscr{C}$ of first-order formulas is a reduction class with respect to satisfiability.*

PROOF. Let $F$ be an arbitrary first-order formula. Then $F$ is satisfiable iff $F$ is not a tautology. But, by Theorem 1, there is a member $F_1$ of $\mathscr{D}$, effectively computable from $F$, which is not a tautology iff $\neg F$ is not a tautology. But then $F$ is satisfiable iff $\neg F_1$ is satisfiable and since $\neg F_1$ is in $\mathscr{C}$ this completes the proof.

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