Cutting a tree with Subgraph Complementation is hard, except for some small trees.

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Abstract

For a property Π, Subgraph Complementation to Π is the problem to find whether there is a subset S of vertices in the input graph G such that modifying G by complementing the subgraph induced by S results in a graph satisfying the property Π. We prove that, the problem of Subgraph Complementation to T-free graphs is NP-Complete, for T being a tree, except for 40 trees of at most 13 vertices (a graph is T-free if it does not contain any induced copies of T). Further, we prove that these hard problems do not admit any subexponential-time algorithms, assuming the Exponential Time Hypothesis. As an additional result, we obtain that Subgraph Complementation to paw-free graphs can be solved in polynomial-time.

Keywords: Subgraph Complementation, Graph Modification, Trees, Paw

1 Introduction

A graph property is hereditary if it is closed under vertex deletion. It is well known that every hereditary property is characterized by a minimal set of forbidden induced subgraphs. For example, for chordal graphs, the forbidden set is the set of all cycles, for split graphs, the forbidden set is $\{2K_2, C_4, C_5\}$, for cluster graphs it is $\{P_3\}$, and for cographs it is $\{P_4\}$. The study of structural and algorithmic aspects of hereditary graph classes is central to theoretical computer science.

A hereditary property is called H-free if it is characterized by a singleton set $\{H\}$ of forbidden subgraph. Such hereditary properties are very interesting for their rich structural and algorithmic properties. For example, triangle-free graphs could be among the most studied graphs classes. There is a rich structure theorem for claw-free graphs [1] and it is known that the Independent Set problem is polynomial-time solvable for claw-free graphs [2]. There is a long list of papers on the Independent Set problem on H-free graphs (for example, see [3–5]). Further, there are long standing open problems, such as Erdős-Hajnal conjecture, related to H-free graphs.

We study a graph recognition problem, known as *subgraph complementation* related to H-free graphs. A *subgraph complement* of a graph G is a graph G' obtained from G by flipping the adjacency of pairs of vertices of a set $S \subseteq V(G)$. The operation is known as *subgraph complementation* and is denoted by $G' = G \oplus S$. The operation was introduced by Kamiński et al. [6] in relation with clique-width of a graph. For a class $\mathcal{G}$ of graphs, *subgraph complementation to* $\mathcal{G}$ is the problem to check whether there is a set of vertices S in the input graph G such that $G \oplus S \in \mathcal{G}$. The set of all subgraph complements of graphs in $\mathcal{G}$ is denoted by $\mathcal{G}^{(1)}$. Note that the problem subgraph complementation to $\mathcal{G}$ is a recognition problem of $\mathcal{G}^{(1)}$. A systematic study of this problem has been started by Fomin et al. [7]. They obtained polynomial-time algorithms for this problem for various classes of graphs including triangle-free graphs and P_4-free graphs. A superset of the authors of this paper studied it further [8] and settled the complexities of this problem (except for a finite number of cases) when $\mathcal{G}$ is H-free, for H being a complete graph, a path, a star, or a cycle. They proved that subgraph complementation to H-free graphs is polynomial-time solvable if H is a clique, NP-Complete if H is a path on at least 7 vertices, or a star graph on at least 6 vertices, or a cycle on at least 8 vertices. Further, none of these hard problems admit subexponential-time algorithms, assuming the Exponential-Time Hypothesis.

We study subgraph complementation to H-free graphs, where H is a tree. We come up with a set $\mathcal{T}$ of 40 trees of at most 13 vertices such that if $T \notin \mathcal{T}$, then subgraph complementation to T-free graphs is NP-Complete. Further these hard problems do not admit subexponential-time algorithm, assuming the Exponential-Time Hypothesis. These 40 trees include some paths, stars, bistars (trees with 2 internal vertices), tristars (trees with 3 internal vertices), and some subdivisions of claw. Among these, for four paths (P_ℓ, for $1 \le \ell \le 4$), the problem is known to be polynomial-time solvable. So, our result leaves behind only 36 open cases. Additionally, we prove that the problem is hard when H is a 5-connected prime non-self-complementary graph with at least 18 vertices. As a separate result, we obtain that the problem can be solved in polynomial-time when H is a paw (the unique connected graph on 4 vertices having a single triangle).

2 Preliminaries

In this section, we provide various definitions, notations, and terminologies used in this paper.

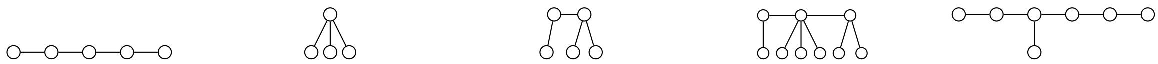
Graphs. For a graph G, the vertex set and edge set are denoted by $V(G)$ and $E(G)$ respectively. A graph G is *H-free* if it does not contain H as an induced subgraph. We denote $\mathcal{F}(H)$ as the family of H-free graphs. The vertex connectivity, $\mathcal{K}(G)$, of a graph G is the minimum number of vertices in G whose removal causes G either disconnected or reduces G to a graph with only one vertex. A graph G is said to be k-connected, if $\mathcal{K}(G) \ge k$. We denote the well-known graphs such as a complete graph, an independent set, a star, a cycle, and a path, each of them on n vertices, by K_n, nK_1, $K_{1,n-1}$, C_n, and P_n respectively.

A graph G, which is isomorphic to its complement $\overline{G}$ is called *self-complementary* graph. If G is not isomorphic to $\overline{G}$, then it is called *non-self-complementary.* The *join* of two graphs G and H, denoted by $G \times H$, is a graph in which each vertex in G is adjacent to all vertices in H. By $G + H$, we denote the *disjoint union* of two graphs G and H. Similarly by rG, we denote the disjoint union of r copies of a graph G. By $G[H]$, we denote the graph obtained from G by replacing each vertex of G with H. That is, $V(G[H]) = V(G) \times V(H)$, and $E(G[H]) = \{((u,v)(u',v'))|(u,u') \in E(G)$ or $(u = u'$and $(v,v') \in E(H)\}$. We denote $G - X$ as the graph obtained from G by removing the vertices in X.

The open neighborhood of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of all the vertices adjacent to v, i.e., $N(v) := \{w \mid vw \in E(G)\}$, and the closed neighborhood of v, denoted by $N[v]$, is defined as $N(v) \cup \{v\}$. A pair of independent vertices in a graph G are called *false-twins*, if they have the same neighborhood in G. Let u be a vertex and X be a vertex subset of G. Let $N_X(u)$ and $N_{\overline{X}}(u)$ denote the neighborhood of u inside the sets X and $V(G) \setminus X$, respectively. We extend the notion of adjacency to set of vertices as: two sets A and B of vertices of G are *adjacent* (resp., *non-adjacent*) if each vertex of A is adjacent (resp., non-adjacent) to each vertex of B. Let G' be a graph and $u \in V(G')$. We say that a graph H is obtained from H' by *vertex duplication*, if H is obtained from H' by replacing each vertex in H' by an independent set of size $r_i \ge 1$. A set of vertices R is said to be *untouched* by a set S in $V(G)$, if $R \cap S = \emptyset$.

A *tree* is an acyclic graph, and the disjoint union of trees is called a *forest.* The *internal tree* T' of a tree T is a tree obtained by removing all the leaves of T. The *center* of a star graph $K_{1,x}$ is the vertex which is connected to all the leaves of $K_{1,x}$. A *bistar* graph $T_{x,y}$, for $x \ge 1$ and $y \ge 1$, is a graph obtained by joining the centers a and b of two star graphs $K_{1,x}$ and $K_{1,y}$ respectively, where a is the x-center (the vertex adjacent to x leaves) and b is the y-center (the vertex adjacent to y leaves) of $T_{x,y}$. Similarly, *tristar* graph $T_{x,y,z}$, for $x \ge 1$, $y \ge 1$ and $z \ge 1$, is a graph obtained by joining the centers a, b, and c of three star graphs $K_{1,x}$, $K_{1,y}$, and $K_{1,z}$ respectively in such a way that a, b, c induce a $K_{1,2}$ with b as the center. The *subdivision of claw*, denoted by $C_{x,y,z}$ for $x \ge 1, y \ge 1, z \ge 1$, is a graph obtained from the claw, $K_{1,3}$, by subdividing its three edges $x-1$ times, $y-1$ times, and $z-1$ times respectively. Some examples are given in Figure 1.

Modular decomposition. A vertex subset X of G is a *module* if $N_{\overline{X}}(u) = N_{\overline{X}}(v)$ for all $u, v \in X$.



(a) P_5 (b) $K_{1,3}$ (c) $T_{1,2}$ (d) $T_{1,3,2}$ (e) $C_{1,2,3}$

Figure 1: Some examples of trees

The *trivial* modules of a graph G are $\emptyset, V(G)$, and all the singletons $\{v\}$ for $v \in V(G)$. A graph is *prime* if it has at least 3 vertices and all its modules are trivial, and *nonprime* otherwise. A nontrivial module M is a *strong module* of a graph G if for every other module M' in G, if $M \cap M' \neq \emptyset$, then either $M \subseteq M'$ or $M' \subseteq M$. A module which induces an independent set is called *independent module* and a module which induces a clique is called *clique module*. Let G be a nonprime graph such that both G and $\overline{G}$ are connected graphs. Then there is a unique partitioning $\mathcal{P}$ of $V(G)$ into maximal strong modules. The quotient graph Q_G of G has one vertex for each set in $\mathcal{P}$ and two vertices in Q_G are adjacent if and only if the corresponding modules are adjacent in G. The modular decomposition theorem due to Gallai [9] says that Q_G is a prime graph. We refer to [10] for more details on modular decomposition and related concepts.

Boolean satisfiability problems. In a 3-SAT formula, every clause contains exactly three literals of distinct variables and the objective of the 3-SAT problem is to find whether there exists a truth assignment which assigns TRUE to at least one literal per clause. The problem is among the first known NP-Complete problems. The Exponential-Time Hypothesis (ETH) and the Sparsification Lemma imply that 3-SAT cannot be solved in subexponential-time, i.e., in time $2^{o(n+m)}$, where n is the number of variables and m is the number of clauses in the input formula. To prove that a problem does not admit a subexponential-time algorithm, it is sufficient to obtain a linear reduction from a problem known not to admit a subexponential-time algorithm, where a linear reduction is a polynomial-time reduction in which the size of the resultant instance is linear in the size of the input instance. All our reductions are trivially linear and we may not explicitly mention the same. We refer to the book [11] for a detailed description.

In a k-SAT formula, every clause contains exactly k literals. The objective of the k-SAT$_{\geq 2}$ problem is to find whether there is a truth assignment for the input k-SAT formula such that at least two literals per clause are assigned TRUE. For every $k \geq 4$, there are two simple linear reductions from 3-SAT to 4-SAT$_{\geq 2}$ and then to k-SAT$_{\geq 2}$ to prove the hardness of k-SAT$_{\geq 2}$. Replace every clause $(\ell_1 \vee \ell_2 \vee \ell_3)$ in the input Φ of 3-SAT by a clause $(\ell_1 \vee \ell_2 \vee \ell_3 \vee x_1)$ - this makes sure that Φ is satisfiable if and only if there is a truth assignment which assigns TRUE to at least two literals per clause of the new formula. A linear reduction from 4-SAT$_{\geq 2}$ to k-SAT$_{\geq 2}$ is also trivial: Replace every clause $(\ell_1 \vee \ell_2 \vee \ell_3 \vee \ell_4)$ in the input Φ of 4-SAT$_{\geq 2}$ by 2^{k-4} clauses each of them contains $\ell_1, \ell_2, \ell_3, \ell_4$, and either the positive literal or the negative literal of $k-4$ new variables $x_1, x_2, \ldots, x_{k-4}$. The 2^{k-4} clauses are to make sure that all combinations of the negative and positive literals of the new variables are present which makes sure that Φ is satisfiable (with two true literal per clause) if and only if the new formula is satisfiable (with two true literals per clause). Since k is a constant, this reduction is a linear reduction.

Proposition 2.1 (folklore)**.** *k-SAT$_{\geq 2}$ is NP-Complete. Further, the problem cannot be solved in time $2^{o(n+m)}$, assuming the ETH.*

By $G \oplus S$, for a graph G and $S \subseteq V(G)$, we denote the graph obtained from G by flipping the adjacency of pairs of vertices in S. The problem that we deal with in this paper is given below.

SC-TO-$\mathcal{F}(H)$: Given a graph G, find whether there is a set $S \subseteq V(G)$ such that $G \oplus S$ is H-free.

We make use of the following known results.

Proposition 2.2 ([8])**.** *Let T be a path on at least 7 vertices. Then* SC-TO-$\mathcal{F}(T)$ *is NP-Complete. Further, the problem cannot be solved in time $2^{o(|V(G)|)}$, unless the ETH fails.*

Proposition 2.3 ([8])**.** *Let T be a star on at least 6 vertices. Then* SC-TO-$\mathcal{F}(T)$ *is NP-Complete. Further, the problem cannot be solved in time $2^{o(|V(G)|)}$, unless the ETH fails.*

We say that two problems A and B are polynomially equivalent, if there is a polynomial-time reduction from A to B and there is a polynomial-time reduction from B to A.

Proposition 2.4 ([8]). *SC-TO-$\mathcal{F}(H)$ and SC-TO-$\mathcal{F}(\overline{H})$ are polynomially equivalent.*

3 Reductions for general graphs

In this section, we introduce two reductions which will be used in the next section to prove hardness for SC-TO-$\mathcal{F}(H)$, when H is a tree. We believe that these reductions will be very useful in an eventual dichotomy for the problem for general graphs H. The first reduction is a linear reduction from SC-TO-$\mathcal{F}(H')$ to SC-TO-$\mathcal{F}(H)$ where H is obtained from H' by vertex duplication. The second reduction proves that for every 5-connected non-self-complementary prime graph H with a clique or independent set of size 4, SC-TO-$\mathcal{F}(H)$ is NP-Complete and does not admit a subexponential-time algorithm, assuming the ETH.

3.1 Graphs with duplicated vertices

Here, with the help of a linear reduction, we prove that the hardness results for a prime graph H' translates to that for H, where H is obtained from H' by vertex duplication.

Lemma 3.1. *Let H' be a prime graph with vertices $V(H') = \{v_1, v_2, \ldots, v_t\}$. Let H be a graph obtained from H' by replacing each vertex v_i in H' by an independent set I_i of size r_i, for some integer $r_i \geq 1$. Then there is a linear reduction from SC-TO-$\mathcal{F}(H')$ to SC-TO-$\mathcal{F}(H)$.*

Let H' and H be graphs mentioned in Lemma 3.1. Let r be the maximum integer among the r_is, i.e., $r = \max_{i=1}^{i=t} r_i$. We note that H' is the quotient graph of H. See Figure 2 for an example.

Construction 1. *Given a graph G' and an integer $r \geq 1$, the graph G is constructed from G' as follows: for each vertex u of G', replace u with a set W_u which induces an rK_r. The so obtained graph is G (see Figure 3 for an example).*

(a) H' (b) H

Figure 2: An example of H' and H. Here, $r_1 = r_2 = r_4 = 1, r_3 = r_5 = 2$, and $r = 2$, assuming an order of vertices of H' from left to right.

(a) G' (b) G

Figure 3: An example of Construction 1 for a graph G' isomorphic to P_7, and for an interger $r = 2$. The lines connecting two circles (bold or dashed) indicate that the vertices corresponding to that circles are adjacent.

Lemma 3.2. *If $G' \oplus S' \in \mathcal{F}(H')$ for some $S' \subseteq V(G')$, then $G \oplus S \in \mathcal{F}(H)$, where S is the union of vertices in W_u for every vertex $u \in S'$.*

Proof. Let an H be induced by A (say) in $G \oplus S$. Recall that G is constructed by replacing each vertex u in G' with a module W_u which induces an rK_r. If $A \subseteq W_u$ for some vertex u in G', then H is an induced subgraph of either rK_r (if $u \notin S'$) or $\overline{rK_r}$ (if $u \in S'$). Then H', the quotient graph of H, is either an independent set or a complete graph. This is not true as H' is a prime graph. Therefore, A has nonempty intersection with more than one W_us. For a vertex u in G', either W_u is a subset of S

(if $u \in S'$) or W_u has empty intersection with S (if $u \notin S$). Therefore, if A has nonempty intersection with W_u, then $A \cap W_u$ is a module of the H induced by A. Therefore, $A \cap W_u \subseteq I_i$ for some $1 \leq i \leq t$. Let U_i be the set of vertices u in G' such that I_i (in the H induced by A) has a nonempty intersection with W_u. Arbitrarily choose one vertex from U_i. Let A' be the set of such chosen vertices for all $1 \leq i \leq t$. We claim that A' induces an H' in $G' \oplus S'$. Let u_i and u_j be the vertices chosen for I_i and I_j respectively, for $i \neq j$. Since $A \cap W_{u_i} \subseteq I_i$ and $A \cap W_{u_j} \subseteq I_j$, and $i \neq j$, we obtain that $u_i \neq u_j$. It is enough to prove that u_i and u_j are adjacent in $G' \oplus S'$ if and only if v_i and v_j are adjacent in H'. If u_i and u_j are adjacent in $G' \oplus S'$, then W_{u_i} and W_{u_j} are adjacent in $G \oplus S$. This implies that I_i and I_j are adjacent in H. Hence v_i and v_j are adjacent in H'. For the converse, assume that v_i and v_j are adjacent in H'. This implies that I_i and I_j are adjacent in H. Therefore, W_{u_i} and W_{u_j} are adjacent in $G \oplus S$. Hence u_i and u_j are adjacent in $G' \oplus S'$. This completes the proof. □

Lemma 3.3. *If $G \oplus S \in \mathcal{F}(H)$ for some $S \subseteq V(G)$, then $G' \oplus S' \in \mathcal{F}(H')$, where S' is a subset of vertices of G' obtained in such a way that whenever all vertices of a K_r from a module W_u (which induces an rK_r) is in S, then the corresponding vertex u in G' is included in S'.*

Proof. Suppose $G' \oplus S'$ contains an H' induced by a set $A' = \{v_1, v_2, \ldots, v_t\}$. If a vertex u in G' is in S', then all vertices of a K_r from W_u is in S. Therefore, there is an independent set of size r in $W_u \cap S$. Similarly, if $u \notin S'$, then there is an independent set of size r in $W_u \setminus S$ formed by one vertex, which is not in S, from each copy of K_r in W_u which is not in S. We construct A as follows: for each vertex $v_i \in A'$, if $v_i \in S'$, include in A an independent set $I_i \subseteq W_{v_i} \cap S$ such that $|I_i| = r_1$, and if $v_i \notin S'$, include in A an independent set $I_i \subseteq W_{v_i} \setminus S$ such that $|I_i| = r_i$. We claim that A induces an H in $G \oplus S$. Note that each chosen I_i is a module in $G \oplus S$. Since $I_i \subseteq S$ if and only if $v_i \in S'$, we obtain that I_i and I_j are adjacent in $G \oplus S$ if and only if v_i and v_j are adjacent in the H' induced by A'. This completes the proof. □

Lemma 3.1 follows directly from Lemma 3.2 and 3.3. When the lemma is applied on trees, we get the following corollary. We note that the quotient tree Q_T of a tree is prime if and only if T is not a star graph - by our definition, a prime graph has at least 3 vertices.

Corollary 3.4. *Let T be a tree which is not a star graph, and let Q_T be its quotient tree. Then there is a linear reduction from* SC-TO-$\mathcal{F}(Q_T)$ *to* SC-TO-$\mathcal{F}(T)$.

3.2 5-connected graphs

Here, we obtain hardness results for SC-TO-$\mathcal{F}(H)$, where H is a 5-connected graphs satisfying some additional constraints.

Theorem 3.5. *Let H be a 5-connected, non self-complementary, prime graph with an independent set of size 4 or with a clique of size 4. Then* SC-TO-$\mathcal{F}(H)$ *is NP-Complete. Further, the problem cannot be solved in time $2^{o(|V(G)|)}$, unless the ETH fails.*

We have the following corollary from the fact that the Ramsey number $R(4, 4) = 18$.

Corollary 3.6. *Let H be a 5-connected, non self-complementary, prime graph with at least 18 vertices. Then* SC-TO-$\mathcal{F}(H)$ *is NP-Complete. Further, the problem cannot be solved in time $2^{o(|V(G)|)}$, unless the ETH fails.*

Let H be a 5-connected graph satisfying the constraints mentioned in Theorem 3.5. Let H has t vertices and let $V' \subseteq V(H)$ induces either a K_4 or a $4K_1$ in H. We use Construction 2 for a reduction from 4-SAT$_{\geq 2}$ to prove Theorem 3.5.

Construction 2. *Let Φ be a* 4-SAT *formula with n variables $X_1, X_2, \cdots, X_n$, and m clauses $C_1, C_2, \cdots,$ C_m. We construct the graph G_Φ as follows.*

- *For each variable X_i in Φ, the variable gadget also named as X_i consists of the union of two special sets $X_{i1} = \{x_i\}$ and $X_{i2} = \{\overline{x_i}\}$, and $t - 2$ other sets $X_{i3}, X_{i4} \ldots X_{it}$ such that each X_{ij}, for $3 \leq j \leq t$ induces an $\overline{H}$. Make the adjacency between these X_{ij}s in such a way that taking* *one vertex each from these sets induces an H, where X_{i1} and X_{i2} correspond to two non-adjacent vertices, if V' forms a K_4, and correspond to two adjacent vertices, if V' forms a $4K_1$. If V' forms a clique then add an edge between X_{i1} and X_{i2}, and if V' forms an independent set, then remove the edge between X_{i1} and X_{i2}. The vertices x_is and $\overline{x}_i$s are called literal vertices denoted by a set L, which induces a clique, if V' is a clique, and induces an independent set, if V' is an independent set.*

- *For each clause C_i of the form $(\ell_{i1} \vee \ell_{i2} \vee \ell_{i3} \vee \ell_{i4})$ in Φ, the clause gadget also named as C_i consists of $t-4$ copies of $\overline{H}$ denoted by C_{ij}, for $1 \leq j \leq (t-4)$. Let the four vertices introduced (in the previous step) for the literals $\ell_{i1}, \ell_{i2}, \ell_{i3}$, and ℓ_{i4} be denoted by $L_i = \{y_{i1}, y_{i2}, y_{i3}, y_{i4}\}$. The adjacency among each of these C_{ij}s and the literal vertices L_i is in such a way that, taking one vertex from each C_{ij}s and the vertices in L_i induces an H.*

This completes the construction.

An example of the construction is shown in Figure 5 for a graph H given in Figure 4. Keeping a module isomorphic to $\overline{H}$ guarantees that not all vertices in the module is present in a solution S of G_Φ (i.e., $G_\Phi \oplus S$ is H-free). The purpose of variable gadget X_i is to make sure that both x_i and $\overline{x_i}$ are not placed in a solution S, so that we can assign TRUE to all literals corresponding to literal vertices placed in S, to get a valid truth assignment for Φ. On the other hand, any truth assignment assigning TRUE to at least two literals per clause makes sure that the set S formed by choosing literal vertices corresponding to TRUE literals destroys copies of H formed by clause gadgets C_i and the corresponding sets L_i of literal vertices.

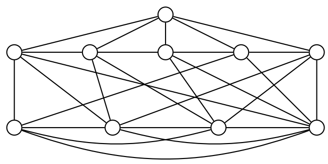


Figure 4: An example of a 5-connected, non-self-complementary, prime graph with a K_4 (formed by the lower four vertices)

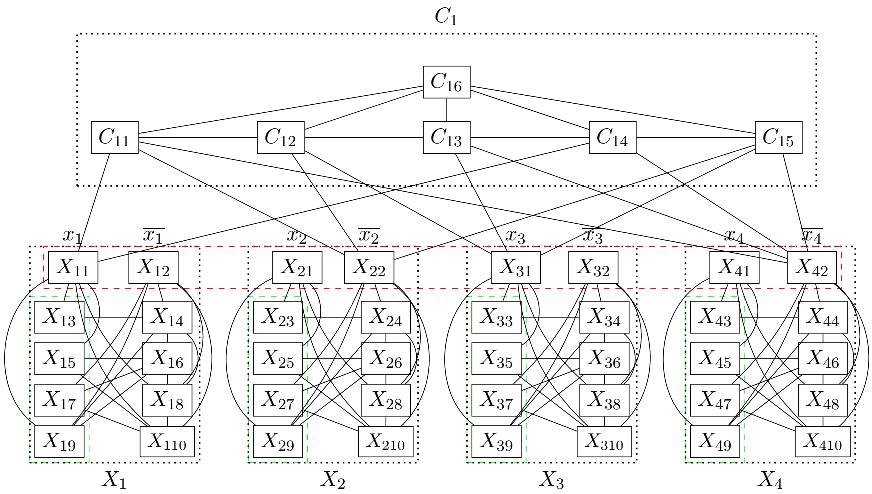


Figure 5: An example of Construction 2 for the formula $\Phi = C_1$ where $C_1 = x_1 \vee \overline{x_2} \vee x_3 \vee \overline{x_4}$ corresponding to the graph H shown in Figure 4 with a K_4. The lines connecting two rectangles indicate that each vertex in one rectangle is adjacent to all vertices in the other rectangle. If there is no line shown between two rectangles, then the vertices in them are non-adjacent, with the exceptions – (i) all the vertices in a red rectangle (dashed) together form a clique; (ii) the rectangles in each green rectangle (dashed) are adjacent.

Lemma 3.7. *Let Φ be a yes-instance of* 4-SAT$_{\geq 2}$ *and ψ be a truth assignment satisfying Φ. Then $G_\Phi \oplus S$ is H-free where S is the set of literal vertices whose corresponding literals were assigned TRUE by ψ.*

Proof. Let $G_\Phi \oplus S$ contain an H induced by A (say). Since H is a prime graph and $\overline{H}$ is not isomorphic to H, $A \cap Y \leq 1$ where Y is a module isomorphic to $\overline{H}$. Thus, $\mid A \cap X_{ij} \mid$ is at most one. Therefore, since $\{x_i, \overline{x_i}\}$ is not a subset of S, we obtain that X_i does not have an induced H in $G_\Phi \oplus S$. Recall that, the vertices in X_{ij} (for $3 \leq j \leq t$) are non-adjacent to $V(G) \setminus X_i$, and H is 5-connected. This implies that $A \cap (X_i \setminus \{x_i, \overline{x_i}\}) = \emptyset$.

Since C_i contains $t-4$ sets of $\overline{H}$s, $|C_i \cap A| \leq t-4$. Now assume that A contains vertices from two clause gadgets C_i and C_j. Since the vertices in C_i are only adjacent to the four literal vertices corresponding to the clause C_i, and H is 5-connected, removing the four literal vertices corresponding to C_i disconnects the graph which is not possible –note that C_i and C_j are non-adjacent. Hence, A contains vertices from at most one clause gadget C_i.

Note that L induces a $K_n \times nK_1$ in $G_\Phi \oplus S$, if V' induces a clique, and induces a $K_n + nK_1$ in $G_\Phi \oplus S$, if V' induces an independent set. Therefore, H is not an induced subgraph of the graph induced by L in $G_\Phi \oplus S$. Recall that the vertices in $A \cap C$ are from at most one clause gadget C_i, and at most one vertex from each of the sets C_{ij} in C_i is in $A \cap C_i$. We know that C_i is non-adjacent to all literal vertices corresponding to the literals not in the clause C_i, and H is 5-connected. Therefore, $A \cap L = \{y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}\}$. Since at least two vertices in $A \cap L$ is in S, the graph induced by A in $G \oplus S$ is not isomorphic to H. ☐

Lemma 3.8. *Let Φ be an instance of* 4-SAT$_{\geq 2}$*. If $G_\Phi \oplus S$ is H-free for some $S \subseteq V(G_\Phi)$, then there exists a truth assignment satisfying Φ.*

Proof. Let $G_\Phi \oplus S$ be H-free for some $S \subseteq V(G_\Phi)$. We want to find a satisfying truth assignment of Φ. Since each of the C_{ij}s in C_i, for $1 \leq i \leq m$ and $1 \leq j \leq t-4$, induces an $\overline{H}$, there is at least one vertex in each C_{ij} which is not in S. Then, if at least two vertices from L_i are not in S, then there is an induced H by vertices in L_i and one vertex each from $C_{ij} \setminus S$, for $1 \leq j \leq t-4$. Therefore, at least two vertices from L_i are in S. Next we prove that $\{x_i, \overline{x_i}\}$ is not a subset of S. For each X_{ij} (for $3 \leq j \leq t$), since each of them induces an $\overline{H}$, at least one vertex is not in S. Then, if both x_i and $\overline{x_i}$ are in S, then there is an H induced by $x_i, \overline{x_i}$, and one vertex each from $X_{ij} \setminus S$, for $3 \leq j \leq t$. Now, it is straight-forward to verify that assigning TRUE to every literal x_i such that $x_i \in S$, is a valid satisfying truth assignment of Φ. ☐

Now, Theorem 3.5 follows from Lemma 3.7 and Lemma 3.8.

4 Trees

By $\mathcal{T}$ we denote the set $\mathcal{P} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{C}$, where $\mathcal{P} = \{P_x \mid 1 \leq x \leq 5\}$, $\mathcal{T}_1 = \{K_{1,x} \mid 1 \leq x \leq 4\}$, $\mathcal{T}_2 = \{T_{x,y} \mid 1 \leq x \leq y \leq 4\}$, $\mathcal{T}_3 = \{T_{1,0,1}, T_{1,0,2}\} \cup \{T_{x,y,z} \mid x = 1, 1 \leq y \leq 4, z \leq 5\}$, and $\mathcal{C} = \{C_{1,1,1}, C_{1,1,2}, C_{1,1,3}, C_{1,2,2}, C_{1,2,3}, C_{2,2,2}, C_{2,2,3}\}$. These sets denote the paths, stars, bistars, tristars, and subdivisions of claw not handled by our reductions.

We note that $|\mathcal{P}| = 5$, $|\mathcal{T}_1| = 4$, $|\mathcal{T}_2| = 10$, $|\mathcal{T}_3| = 22$, and $|\mathcal{C}| = 7$. But, a star graph $K_{1,x}$ is a path in $\mathcal{P}$ if $x \leq 2$, the bistar graph $T_{1,1}$ is the path P_4, the tristar graphs $T_{1,0,1}$ is P_5, and the subdivision of claw $C_{1,1,1}$ is the star graph $K_{1,3}$, $C_{1,1,2}$ is the bistar graph $T_{1,2}$, $C_{1,1,3}$ is the tristar graph $T_{1,0,2}$, and $C_{1,2,2}$ is the tristar graph $T_{1,1,1}$. Therefore, $|\mathcal{T}| = 40$, and the tree of maximum order in $\mathcal{T}$ is $T_{1,4,5}$ with 13 vertices. We prove the following theorem in this section.

Theorem 4.1. *Let T be a tree not in $\mathcal{T}$. Then* SC-TO-$\mathcal{F}(T)$ *is NP-Complete. Further, the problem cannot be solved in time $2^{o(|V(G)|)}$, unless the ETH fails.*

This task is achieved in seven sections. In the first section, we prove that there is a linear reduction from SC-TO-$\mathcal{F}(T')$ to SC-TO-$\mathcal{F}(T)$, where T is a prime tree and T' is its internal tree. In the second section, we deal with trees with at least 4 leaves and at least 3 internal vertices, and satisfying some additional constraints. Then in third and fourth sections, we prove the hardness for bistars and tristars respectively, leaving behind a finite number of open cases. Fifth section proves the hardness for P_6, thereby leaving only one unsolved case (P_5) among paths. Sixth section settles subdivions of claw sans a finite number of cases. We combine all these results in seventh section to prove Theorem 4.1.

4.1 Removing leaves

In this section, with a very simple reduction, we prove that the hardness transfers from T' to T, where T is a prime tree and T' is its internal tree. We use Construction 3 for the reduction. See Figure 6, for an example of T and T'.

Lemma 4.2. *Let T be a prime tree and let T' be its internal tree. Then there is a linear reduction from* SC-TO-$\mathcal{F}(T')$ *to* SC-TO-$\mathcal{F}(T)$.

(a) T' (b) T

Figure 6: An example of T' and T

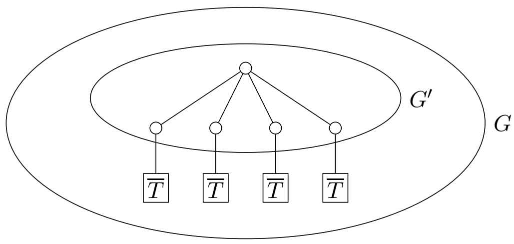


Figure 7: An example of Construction 3

Construction 3. *Let (G', T) be the input to the construction, where G' is a graph and T is a prime tree. The graph G is constructed from G' as follows: for every vertex u of G', introduce a $\overline{T}$, denoted by W_u in the neighbourhood of u. (See Figure 7 for an example.)*

Lemma 4.3. *If $G' \oplus S' \in \mathcal{F}(T')$ for some $S' \subseteq V(G')$, then $G \oplus S' \in \mathcal{F}(T)$.*

Proof. Let a T be induced by a set A in $G \oplus S'$. Note that T is a prime graph and $\overline{T}$ is not isomorphic to T. Thus, W_u does not induce a T. For any vertex $v \in W_u$, the only neighbor of v in $V(G) \setminus W_u$ is u. Hence $A \cap G'$ is nonempty. Let $u \in G'$ be a vertex in A. Recall that the vertices in W_u are the only neighbors of u in $V(G) \setminus V(G')$. Since T is a prime tree and W_u induces a module in G, $|A \cap W_u| \leq 1$. Thus, $A \cap W_u$ cannot contain any internal vertex of T which implies that $G' \oplus S'$ contains a T'. However, that is not possible according to the statement of the lemma. $\square$

Lemma 4.4. *If $G \oplus S \in \mathcal{F}(T)$ for some $S \subseteq V(G)$, then $G' \oplus S' \in \mathcal{F}(T')$, where $S' = S \cap V(G')$.*

Proof. If $G' \oplus S'$ contains a T' induced by A (say), then $G \oplus S$ will contain a T unless for at least one vertex $u \in A$, all vertices of W_u belong to S. However, in that case we will have a T induced by W_u in $G \oplus S$, which is a contradiction. $\square$

Now, Lemma 4.2 follows from Lemma 4.3 and Lemma 4.4.

4.2 Trees with at least 4 leaves and 3 internal vertices

In this section, we prove hardness results for SC-TO-$\mathcal{F}(T)$, when T is a tree with at least 4 leaves and at least 3 internal vertices, and satisfying some additional constraints. The reduction is from k-SAT$_{\geq 2}$.

Theorem 4.5. *Let T be a tree with at least 4 leaves and at least 3 internal vertices. Let T' be the internal tree of T. Assume that the following properties are satisfied.*

(i) If T' is a star graph, then at least one of the following conditions are satisfied:

(a) every leaf of T' has at least two leaves of T as neighbors, or

(b) the center of the star T' has no leaf of T as neighbor, or

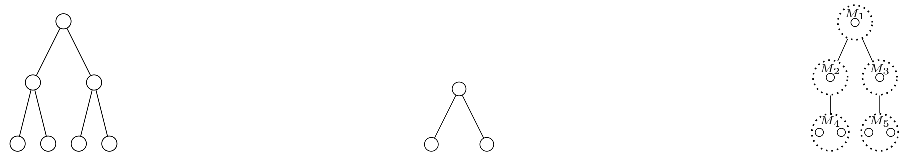
(c) T is either a $C_{1,2,2,2,}$ or a $C_{1,2,2,2,2}$.

(ii) There are no two adjacent vertices of degree 2 in T such that neither of them is adjacent to any leaf of T.

Then SC-TO-$\mathcal{F}(T)$ *is NP-Complete. Further, the problem cannot be solved in time $2^{o(|V(G)|)}$, unless the ETH fails.*

Let T be a tree and T' be its internal tree. Assume that T satisfies the conditions of Theorem 4.5. Let T has p internal vertices and k leaves. Then, T has $t = p + k$ vertices, and T' has p vertices. Let $V(T) = \{v_1, v_2, \ldots, v_t\}$, where $\{v_1, v_2, \ldots, v_p\}$ forms the internal vertices. Without loss of generality, we assume that v_1 and v_2 are adjacent. Let $\mathcal{M} = \{M_1, M_2, \ldots, M_{t'}\}$ be the modular decomposition of T, where $t' = k' + p$, where k' is the number of modules containing leaves. Let $\{M_1, M_2, \ldots, M_p\}$ forms the modules ($M_i = \{v_i\}$, for $1 \leq i \leq p$) containing the internal vertices, and $\{M_{p+1}, M_{p+2}, \ldots, M_{t'}\}$ forms the modules containing leaves. Let r be the maximum size of modules containing leaves, i.e., $r = max_{i=p+1}^{i=t'}\{|M_i|\}$. Let Q_T be the quotient tree of T. By R, we denote the graph $\overline{Q_T}[rK_r]$, i.e., R is the graph obtained from $\overline{Q_T}$ by replacing each vertex by a disjoint union of r copies of K_r.

We use Construction 4 for the reduction from k-SAT$_{\geq 2}$. The reduction is very similar to the reduction used to handle 5-connected graphs in Section 3.2.



(a) T (b) T' (c) Q_T

Figure 8: An example of a Tree T (8a) which satisfies the properties of Theorem 4.5, its internal tree T' (8b), and its quotient graph Q_T (8c) in which five modules of T are represented with circles (dotted). The bold circles represent the vertices in T. The lines connecting two circles (dotted) indicate that each vertex in one circle is adjacent to all vertices in the other circle.

Construction 4. *Let Φ be a k-*SAT *formula with n variables $X_1, X_2, \cdots, X_n$, and m clauses $C_1, C_2, \cdots,$ C_m. We construct the graph G_Φ as follows.*

- *For each variable X_i in Φ, the variable gadget, also named X_i, consists of two special sets $X_{i1} = \{x_i\}, X_{i2} = \{\overline{x_i}\}$, and $t' - 2$ other sets $X_{i3}, X_{i4}, \ldots, X_{it'}$, where each of the set in $\{X_{i3}, X_{i4}, \ldots, X_{it'}\}$ induces an R. We have $X_i = \bigcup_{j=1}^{j=t'} X_{ij}$. The sets X_{ia} and X_{ib} are adjacent if and only if M_a and M_b are adjacent. We remove the edge between X_{i1} and X_{i2} to end the construction of the variable gadget (recall that v_1 and v_2 are adjacent in T). Let $X = \bigcup_{i=1}^{i=n} X_i$. The vertices x_i and $\overline{x_i}$ are called literal vertices, and L is the set of all literal vertices. The set L forms an independent set of size $2n$.*

- *For each clause C_i in Φ of the form $(\ell_{i1} \vee \ell_{i2} \vee \ell_{i3} \vee \ldots \vee \ell_{ik})$, the clause gadget, also named C_i, consists of p copies of $\overline{T}$s called C_{i1}, $C_{i2}, \ldots C_{ip}$. The set of union of all clause gadgets is denoted by C. Let the k vertices introduced (in the previous step) for the literals $\ell_{i1}, \ell_{i2}, \ell_{i3}, \ldots \ell_{ik}$ be denoted by $L_i = \{y_{i1}, y_{i2}, y_{i3}, \ldots y_{ik}\}$. Make the adjacency among these sets C_{ij}s and the corresponding literal vertices in L_i in such a way that, taking one vertex from each set C_{ij} along with the literal vertices L_i induces a T, where the vertices in L_i correspond to the k leaves of T.*

We observe that C_i is obtained from the internal tree of T where each vertex is replaced by $\overline{T}$. In addition to this, every vertex in C_i is adjacent to all literal vertices corresponding to literals not in C_i.

- *For all $i \neq j$, the set C_i is adjacent to the set C_j.*
- *The vertices in $X_i \setminus \{x_i, \overline{x_i}\}$ are adjacent to $V(G) \setminus X_i$, for $1 \leq i \leq n$.*

  This completes the construction of the graph G_Φ (see Figure 9 for an example).

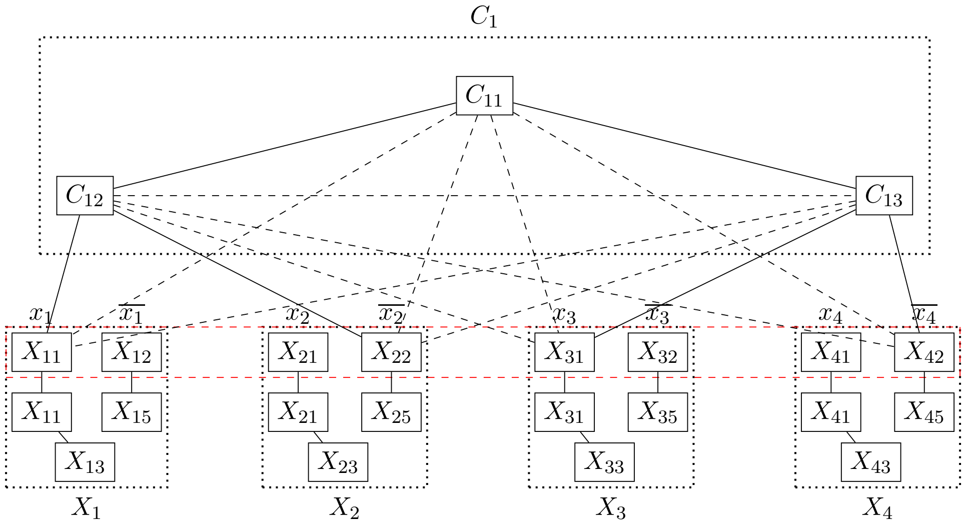


Figure 9: An example of Construction 4 for the formula $\Phi = C_1$ where $C_1 = x_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_4$ corresponding to the tree T shown in Figure 8. The bold lines (respectively dashed lines) connecting two rectangles indicate that each vertex in one rectangle is adjacent (respectively non-adjacent) to all vertices in the other rectangle. If there is no line shown between two rectangles, then the vertices in them are adjacent, with an exception – all the vertices in the red rectangle (dashed) together form an independent set. Similarly, if there is no line shown between two rectangles in the dotted rectangles, then the rectangles in them are non-adjacent.

Observation 4.6 says that T cannot be an induced subgraph of any of the X_{ij}s and any of the C_is, and that any solution of G_Φ leaves an independent set of size r untouched in X_{ij}, which induces an R.

Observation 4.6. *(i) T is not an induced subgraph of R.*

(ii) T is not an induced subgraph of C_i for any $1 \leq i \leq m$.

(iii) Let S be a subset of vertices of G_Φ such that $G_\Phi \oplus S$ is T-free. Then there is an independent set of size r in $X_{ij} \setminus S$ for any $1 \leq i \leq n$ and $3 \leq j \leq t'$.

Proof. To prove (i) by contradiction, assume that A induces a T in $\overline{Q_T}[rK_r]$. Clearly, there is no induced copy of T in rK_r. Therefore, A must have nonempty intersection with more than one copy of rK_r. Since Q_T and $\overline{Q_T}[rK_r]$ have t' modules each, A must have nonempty intersection with every copy of rK_r. Then the quotient graph of the graph induced by A is $\overline{Q_T}$, which is not isomorphic to Q_T, as no nontrivial tree is self-complementary. Therefore, A does not induce T.

To prove (ii) by contradiction, assume that A induces a T in C_i. We recall that C_i represents the internal tree of T, where each vertex is replaced by $\overline{T}$. Since no nontrivial tree is self-complementary, A must have nonempty intersection with at least two sets C_{ij} and $C_{i\ell}$. Then, the graph induced by A has only at most p modules, which contradicts with the fact that T has $t' > p$ modules.

To prove (iii) by contradiction, assume that there is no independent set of size r in X_{ij} untouched by S. This implies that, for every copy of rK_r in X_{ij}, one clique of size r is in S. Let A be a union of such cliques, one from each copy of rK_r. The set A induces $\overline{Q_T}[K_r]$ in G_Φ, and $Q_T[rK_1]$ in $G_\Phi \oplus S$. Since T is an induced subgraph of $Q_T[rK_1]$, we obtain a contradiction. ∎

Before proving the forward direction of the correctness of the reduction, we handle a few cases in the forward direction separately.

Lemma 4.7. *Let Φ be a yes-instance of k-SAT$_{\geq 2}$ and ψ be a truth assignment satisfying Φ. Let S be the set of literal vertices whose corresponding literals were assigned TRUE by ψ. Then there exists no set A such that A induces a T in $G_\Phi \oplus S$, and $A \subseteq C \cup L$ and $|A \setminus (C_i \cup L_i)| = 1$, and $|A \cap C_i| \geq 2$ (for some $1 \leq i \leq n$).*

Proof. Assume for a contradiction that there exists such a set A. Let $A \setminus (C_i \cap L_i) = \{w\}$. Clearly, $A \cap C_i$ is an independent set - otherwise, there is a triangle formed by w and two adjacent vertices in $A \cap C_i$. Further, w is an internal vertex of the tree induced by A. We recall that the independent number of a tree T is at most $|V(T)| - 1$, which is achieved when the tree is a star graph. Since C_i corresponds to the internal tree T' (having p vertices) of T, we obtain that $A \cap C_i$ can have vertices from only $p-1$ modules, say $C_{i1}, C_{i2}, \ldots, C_{ip-1}$ of C_i, and A has nonempty intersection with $p-1$ sets in C_i when T' is a star graph. Since $A \cap L_i$ induces a subgraph of $K_2 + (k-2)K_1$ (at least two vertices of L_i are in S), all the leaves of the tree induced by A cannot be from L_i. Therefore, at least one leaf is from C_i. Since L_i can contribute only one internal vertex, $p-2$ internal vertices of the tree must be from C_i (the remaining internal vertex is w). This implies that A has nonempty intersection with $p-1$ sets say $C_{i1}, C_{i2}, \ldots, C_{ip-1}$ (a leaf and an internal vertex cannot come from a set $C_{i\ell}$, which is a module). Hence the internal tree T' of T is a star graph. Let $x, x' \in A \cap L_i$ be such that x is the internal vertex which is adjacent to x'. The vertex x cannot get a leaf from C_i as w is adjacent to every vertex in C_i. Then x is an internal vertex of the tree having a single leaf (x') of T as neighbor. Further, the center vertex (w) of the internal star has a leaf of T (from C_i) as neighbor. Then, by the assumption in the statement of Theorem 4.5, T is either $C_{1,2,2,2}$ or $C_{1,2,2,2,2}$. Let T be $C_{1,2,2,2}$. Let C_{i1} correspond to the root of the internal tree and C_{i2}, C_{i3}, and C_{i4} correspond to the leaves of the internal tree T' of T. Let y_{i1}, y_{i2}, y_{i3}, and y_{i4} correspond to the leaves of T adjacent to C_{i1}, C_{i2}, C_{i3}, and C_{i4} respectively. Clearly, A contains $c_{i2} \in C_{i2}, c_{i3} \in C_{i3}, c_{i4} \in C_{i4}$, and all vertices in $L_i = \{y_{i1}, y_{i2}, y_{i3}, y_{i4}\}$. Then, all vertices in $A \cap C_i$ are internal vertices of the tree, which is a contradiction. The case when T is $C_{1,2,2,2,2}$ can be handled in a similar way. ◻

Lemma 4.8. *Let Φ be a yes-instance of k-SAT$_{\geq 2}$ and ψ be a truth assignment satisfying Φ. Let S be the set of literal vertices whose corresponding literals were assigned TRUE by ψ. Then there exists no set A such that A induces a T in $G_\Phi \oplus S$, and $A \subseteq C_i \cup C_j \cup L$, and $|A \cap C_i| = |A \cap C_j| = 1$ (for some $1 \leq i \neq j \leq n$).*

Proof. Assume that $A \cap C_i = \{c_i\}$ and $A \cap C_j = \{c_j\}$. Since the rest of the vertices in A are from L, there is at most one internal vertex from $L \cap A$. If there are no internal vertices from $A \cap L$, then T has only at most two internal vertices, a contradiction. Assume that there is one internal vertex from $L \cap A$. Then A induces a tristar graph. Without loss of generatlity, assume that c_i is the center of the internal P_3 and x is the internal vertex from L, and $x' \in L$ be the leaf adjacent to x. Assume that c_i has no attached leaf, i.e., T is the tristar graph $T_{1,0,k-1}$. Since none of the leaves are adjacent to c_i, all k leaves are from L_i, i.e., $L_i \subseteq A$ (recall that c_i is adjacent to all literal vertices correspond to literals not in C_i). This is a contradiction, as there is an edge induced by L_i in $G_\Phi \oplus S$. Therefore, c_i has some attached leaves in the tree induced by A. Hence, by the condition (i) of Theorem 4.5, T is either $C_{1,2,2,2}$ or $C_{1,2,2,2,2}$. These cases give contradictions as then there are more than three internal vertices. ◻

Lemma 4.9. *Let Φ be a yes-instance of k-SAT$_{\geq 2}$ and ψ be a truth assignment satisfying Φ. Let S be the set of literal vertices whose corresponding literals were assigned TRUE by ψ. Then T is not an induced subgraph of the graph induced by $C_i \cup L_i$ in $G_\Phi \oplus S$, for any $1 \leq i \leq n$.*

Proof. Assume that $A \subseteq C_i \cup L_i$ induces a T in $G_\Phi \oplus S$. By Observation 4.6(ii), A is not a subset of C_i. Clearly, $A \cap L_i$ can have at most one edge, as L induces $K_n + nK_1$ in $G_\Phi \oplus S$. No other vertex in $A \cap L_i$ other than the end vertices of this edge can be an internal vertex of the T induced by A (by construction, no vertex in L_i has two modules C_{ij} and $C_{i\ell}$ as neighbors as the vertices in L_i correspond to the leaves of T). Therefore, at least $p-2$ internal vertices are from $A \cap C_i$.

Since L_i cannot contribute all leaves (at least two vertices in L_i are in S), at least one leaf must be from C_i. Therefore, C_i contributes only at most $p-1$ internal vertices. Therefore, there are two vertices $u, v \in L_i \cap A$ such that uv is an edge in $G_\Phi \oplus S$. Assume that C_i contributes exactly $p-1$ internal vertices. Then A has nonempty intersection with all modules C_{ij} in C_i. Then, the edge uv in $A \cap L_i$ along with the path through $A \cap C_i$ from the neighbor of u in $A \cap C_i$ and the neighbor of v in $A \cap C_i$ forms a cycle, which is a contradiction.

Therefore, exactly $p-2$ internal vertices are from C_i. Then, both u and v must be internal vertices. Then, only at most $k-2$ leaves are from $A \cap L_i$. Let C_{ia} and C_{ib} be the two modules in C_i which do not contribute internal leaves. Each of C_{ia} and C_{ib} contributes only at most two leaf vertices ($\overline{T}$ is $3K_1$-free). If both C_{ia} and C_{ib} contribute leaves, then there is a cycle as described in the previous case. Therefore, exactly $k-2$ leaves are from $A \cap L_i$ and two leaves are from one module, say C_{ia}. Let w_1 and w_2 be the leaves contributed by C_{ia}. Assume that v is adjacent C_{ia}. Let u' be the neighbor of u, other than v, in the tree. Let $u' \in C_{i\ell}$. We note that $a \neq \ell$ (otherwise, there is a triangle formed by u, v, and a vertex in $A \cap C_{ia}$). Let T'' be the tree induced by A. By *leaf-degree* of a vertex in a tree, we mean the number of leaves adjacent to that vertex in the tree. The *leaf-degree sequence* of a tree is the non-decreasing sequence of leaf-degrees of vertices of the tree. We claim that there is a mismatch in the leaf-degree sequences of T and T'', which provides a contradiction. We know that the set U containing one vertex each from all modules, except from C_{ia}, C_{ib}, and $C_{i\ell}$, two vertices $w_1, w_2 \in C_{ia}$, $u' \in C_{i\ell}$, and the vertices in L_i induces T'' in $G_\Phi \oplus S$. Further, $L_i \cup U \cup \{u', w_1, c_{ib}\}$ induces a T in G_Φ, where c_{ib} is any vertex in C_{ib}. Every vertex in $U \cup (L_i \setminus \{u, v\})$ has the same leaf-degree in T and T''. The leaf-degree of u' is one less in T'' than that in T (u is not a leaf in T''). The leaf-degree of u is 0 in both T and T''. The leaf-degree of v is 0 in T and 2 in T'' (v is a leaf in T, and is adjacent to 2 leaves - w_1 and w_2 - in T''). The leaf-degree of c_{ia} is 1 in T (v is the only leaf, otherwise there will be a C_4 in T'' induced by w_1, w_2, v, and the other leaf) and w_1 and w_2 have leaf-degree 0 in T''. The leaf-degree of c_{ib} is 0 in T (if it is adjacent to some leaf, then that leaf in L_i has no neighbor in T'', which is not true). This implies that the leaf-degree sequences of T and T'' are not the same (see Figure 10a for an example).

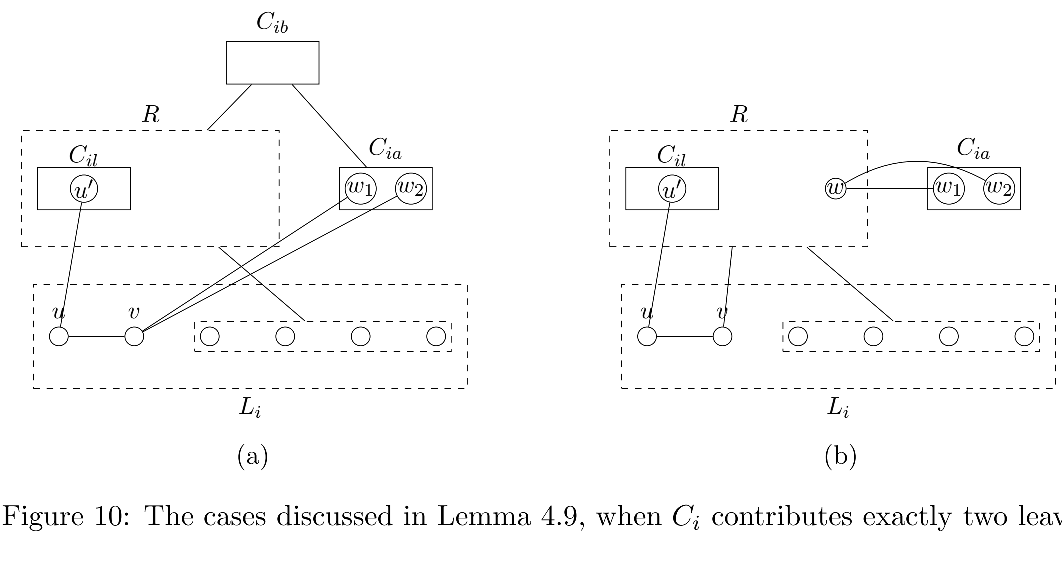


Figure 10: The cases discussed in Lemma 4.9, when C_i contributes exactly two leaves

Therefore, there is a vertex $w \in A \cap C_i$ which is adjacent to the two leaf vertices from C_{ia} (see Figure 10b for an example). Then u and v form two adjacent internal vertices with degree 2 such that neither u nor v is adjacent to a leaf of T, which contradicts with condition (ii) of Theorem 4.5. $\square$

With Lemma 4.7, Lemma 4.8, and Lemma 4.9, we are ready to prove the forward direction of the reduction.

Lemma 4.10. *Let Φ be a yes-instance of k-SAT$_{\geq 2}$ and ψ be a truth assignment satisfying Φ. Then $G_\Phi \oplus S \in \mathcal{F}(T)$ where S is the set of literal vertices whose corresponding literals were assigned TRUE by ψ.*

Proof. Let $G_\Phi \oplus S$ contain a T induced by A (say). We prove the lemma with the help of a set of claims.

Claim 1: A is not a subset of X_i, for $1 \le i \le n$.

Assume that A is a subset of X_i. By Observation 4.6(i), A is not a subset of X_{ij}, for $1 \le j \le t'$. Therefore, A has nonempty intersection with at least two sets X_{ij} and $X_{i\ell}$. Since X_i induces a graph with at most t' modules, and T has t' modules, A has nonempty intersection with all sets X_{ij} ($1 \le j \le t'$). Since $\{x_i, \overline{x_i}\}$ is not a subset of S, we obtain that the quotient graph of the graph induced by A, which is a forest of two trees, is not isomorphic to Q_T, which is a contradiction.

Claim 2: Let $X_i' = X_i \setminus \{x_i, \overline{x_i}\}$ and $\overline{X_i} = V(G_\Phi) \setminus X_i$. If $|A \cap X_i'| \ge 1$, then $A \cap \overline{X_i} = \emptyset$. Similarly, if $|A \cap \overline{X_i}| \ge 1$, then $A \cap X_i' = \emptyset$.

For a contradiction, assume that A contains at least one vertex from X_i' and at least one vertex from $\overline{X_i}$. Since X_i' and $\overline{X_i}$ are adjacent, either $|A \cap X_i'| = 1$ or $|A \cap \overline{X_i}| = 1$.

Assume that $A \cap X_i' = \{u\}$. Since T has at least 3 internal vertices and $(A \cap X_i') \cup (A \cap \overline{X_i})$ induces a star graph, both x_i and $\overline{x_i}$ are in A. Then T is the tristar graph $T_{1,t-5,1}$, which is a contradiction as condition (i) of Theorem 4.5 is not satisfied. Assume that $A \cap \overline{X_i} = \{u\}$. Then with the same argument as given above, we obtain that the graph induced by A is $T_{1,t-5,1}$, which is a contradiction.

Claim 3: A is not a subset of L, the set of all literal vertices.

This follows from the fact that L induces a $K_n + nK_1$ in $G_\Phi \oplus S$.

Claim 4: A cannot have nonempty intersections with three distinct clause gadgets C_i, C_j, and C_ℓ.

Claim 5: There exists no C_i and C_j ($i \ne j$) such that $|A \cap C_i| \ge 2$ and $|A \cap C_j| \ge 2$.

Claim 4 and 5 follow from the fact that C_i and C_j are adjacent for $i \ne j$ and T does have neither a triangle nor a C_4.

Claim 6: A is not a subset of C.

For a contradiction, assume that $A \subseteq C$. By Claim 4, A cannot have nonempty intersections with three distinct clause gadgets C_i, C_j, and C_ℓ. By Observation 4.6, A cannot be a subset of C_i. Therefore, A has nonempty intersection with exactly two clause gadgets C_i and C_j in C. Then A induces a star graph, which is a contradiction as T has at least 3 internal vertices.

Claim 7: If $|A \cap C_i| \ge 2$ and $A \cap C_j \ne \emptyset$ ($i \ne j$), then $(A \cap L) \subseteq L_i$.

Let u be a vertex in $A \cap L \setminus L_i$. Then there is a C_4 formed by u and two vertices in $A \cap C_i$ and one vertex from $A \cap C_j$.

We are ready to prove the lemma. By Claim 1, A is not a subset of X_i. By Claim 2, A cannot have vertices from both $X_i \setminus \{x_i, \overline{x_i}\}$ and $\overline{X_i}$. This implies that $A \subseteq L \cup C$. By Claim 3, A cannot be a subset of L and by Claim 6, A cannot be a subset of C. Therefore, A contains vertices from both L and C. By Claim 4, A cannot have nonempty intersections with three distinct clause gadgets C_i, C_j and C_ℓ. Therefore, $A \cap C \subseteq (C_i \cup C_j)$. Assume that A has nonempty intersection with both C_i and C_j. By Claim 5, we can assume that $|A \cap C_j| = 1$ and $|A \cap C_i| \ge 1$. Assume that $|A \cap C_i| \ge 2$. Then by Claim 7, $(A \cap L) \subseteq L_i$. Then by Lemma 4.7, A cannot induce a T. Let $|A \cap C_i| = |A \cap C_j| = 1$. Then by Lemma 4.8, A cannot induce a T.

Assume that $A \cap C \subseteq C_i$ for some clause gadget C_i. Assume that $A \cap C$ has exactly one vertex. Then the rest of the vertices in A are from L and only one from L can be an internal vertex. Therefore, T has only at most two internal vertices, a contradiction. Therefore, $A \cap C_i$ has at least two vertices. If there are at least two vertices in $A \setminus L_i$, then those two vertices along with two vertices in $A \cap C_i$ forms a C_4. Therefore, $A \cap (L \setminus L_i)$ has at most one vertex. Assume that $|A \cap (L \setminus L_i)| = 1$. Then, by Lemma 4.7, A cannot induce a T. Assume that $A \subseteq C_i \cup L_i$. Then we get a contradiction by Lemma 4.9. ☐

The backward direction of the proof of correctness of the reduction is easy.

Lemma 4.11. *Let Φ be an instance of k-SAT$_{\ge 2}$. If $G_\Phi \oplus S \in \mathcal{F}(T)$ for some $S \subseteq V(G_\Phi)$, then there exists a truth assignment satisfying Φ.*

Proof. Let $G_\Phi \oplus S \in \mathcal{F}(T)$ for some $S \subseteq V(G_\Phi)$. We want to find a satisfying truth assignment of Φ. We know that each of the sets C_{ij}, for $1 \le i \le m$ and $1 \le j \le p$, induces a $\overline{T}$. Therefore, each such set has at least one vertex not in S. Hence at least two vertices in L_i must belong to S, otherwise there is an induced T by vertices in L_i and one vertex each from $C_{ij} \setminus S$, for $1 \le j \le p$.

Similarly, each set X_{ij}, for $1 \leq i \leq n$ and $3 \leq j \leq t'$, induces a $\overline{Q_T}[rK_r]$. By Observation 4.6(iii), there is an independent set of size r untouched by S in X_{ij}. Assume that both x_i and $\overline{x_i}$ are in S. Then there is a copy of $Q_T[rK_1]$ in $G_\Phi \oplus S$, induced by $\{x_i, \overline{x_i}\}$ and one copy of rK_1 from each $X_{ij} \setminus S$ (for $3 \leq j \leq t'$). Since T is an induced subgraph of $Q_T[rK_1]$, we get a contradiction. Therefore, both $\{x_i, \overline{x_i}\}$ is not a subset of S. Now, it is straight-forward to verify that assigning TRUE to each literal corresponding to the literal vertices in S is a satisfying truth assignment for Φ. □

Now, Theorem 4.5 follows from Lemma 4.10 and Lemma 4.11. A special case of tristar graphs comes as a corollary of Theorem 4.5.

Corollary 4.12. *Let x, y, z be integers such that $x \leq z$ and either of the following conditions is satisfied.*

(i) $x = 1, y = 0, z \geq 3$, *or*

(ii) $x \geq 2$

Then SC-TO-$\mathcal{F}(T_{x,y,z})$ *is NP-Complete. Further, the problem cannot be solved in time* $2^{o(|V(G)|)}$, *unless the ETH fails.*

4.3 Bistar graphs

In this section, we prove the hardness for SC-TO-$\mathcal{F}(T)$, where T is a bistar graph $T_{x,y}$, where $y \geq 5$. Recall that, by our convention, $x \leq y$. The reduction is from SC-TO-$\mathcal{F}(K_{1,y})$.

Theorem 4.13. *Let x, y be two integers such that $1 \leq x \leq y$ and $y \geq 5$. Then* SC-TO-$\mathcal{F}(T_{x,y})$ *is NP-Complete. Further, the problem cannot be solved in time* $2^{o(|V(G)|)}$, *unless the ETH fails.*

Lemma 4.14. *Let x, y be two integers such that $1 \leq x \leq y$ and $y \geq 3$. Then there is a linear reduction from* SC-TO-$\mathcal{F}(K_{1,y})$ *to* SC-TO-$\mathcal{F}(T_{x,y})$.

Let $T_{x,y}$ be a bistar graph such that x and y satisfy the constraints mentioned in Lemma 4.14. Clearly, $T_{x,y}$ has $t = x + y + 2$ vertices. Construction 5 is used for the reduction from SC-TO-$\mathcal{F}(K_{1,y})$ to SC-TO-$\mathcal{F}(T_{x,y})$.

Construction 5. *Let (G', x, y) be the input to the construction, where G' is a graph and x and y are integers such that $1 \leq x \leq y$ and $y \geq 3$. Let $t = x + y + 2$. For every vertex u of G', introduce $x + 1$ sets of K_y denoted by $Y_{u_1}, Y_{u_2}, \ldots, Y_{u_{x+1}}$, out of which $Y_{u_{x+1}}$ is in the neighbourhood of u and the sets $Y_{u_1}, Y_{u_2}, \ldots, Y_{u_x}$ are in the neighbourhood of $\overline{Y_{u_{x+1}}}$. Further, for each set Y_{u_i}, for $1 \leq i \leq x$ introduce a set U_{u_i}, which contains $x+2$ sets of $\overline{T_{x,y}}$s denoted by U_{ij} for $1 \leq j \leq x+2$. The adjacency among these sets U_{ij} and Y_{u_i} is in such a way that taking one vertex from each set U_{ij} along with the complement of Y_{u_i} together induces a $T_{x,y}$. Introduce a set of vertices $U_{u_{x+1}}$ which contains $x+1$ copies of $\overline{T_{x,y}}$ denoted by $U_{(x+1)1}, U_{(x+1)2}, \ldots U_{(x+1)(x+1)}$. The edges from $U_{(x+1)j}$s are in such a way that, taking the complement of $Y_{u_{x+1}}$ along with one vertex from Y_{u_x}, and one vertex each from $U_{(x+1)j}$s induces a $T_{x,y}$. Further, make U_{u_x} adjacent to $U_{u_{x+1}}$. Let W_u be the set of all new vertices created for a vertex $u \in V(G')$, i.e., $W_u = \bigcup_{i=1}^{i=x+1}(Y_{u_i} \cup U_{u_i})$. We note that there are no edges between W_u and $W_{u'}$ for two vertices u and u' in G'. This completes the construction of the graph G (see Figure 11 for an example).*

The purpose of U_{u_i} is to make sure that not all vertices in Y_{u_i} is in a solution S of G, so that if at all there is a $K_{1,y}$ induced in $G' \oplus (S \cap V(G'))$, we get a contradiction, as then there will be a $T_{x,y}$ induced in $G \oplus S$ by the vertices in the $K_{1,y}$ and one vertex each, which is not in S, from the Y_{u_i}s.

Lemma 4.15. *If $G' \oplus S' \in \mathcal{F}(K_{1,y})$ for some $S' \subseteq V(G')$, then $G \oplus S' \in \mathcal{F}(T_{x,y})$.*

Proof. Let a $T_{x,y}$ be induced by a set A in $G \oplus S'$. Assume that both the x-center (a vertex adjacent to x leaves) a and the y-center (a vertex adjacent to y leaves) b of the $T_{x,y}$ are from G'. Since each vertex in G' is adjacent to only a clique outside G', at most one leaf of a and at most one leaf of b

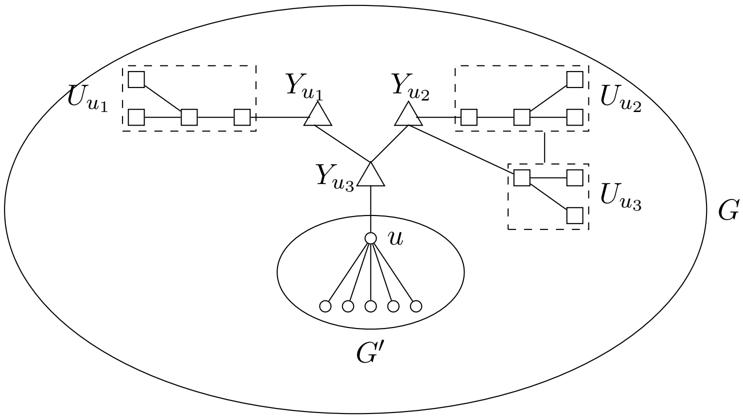


Figure 11: An example of Construction 5 for $x = 2$ and $y = 5$. Each rectangle (bold) represents a $\overline{T_{x,y}}$ and each triangle represents a K_5. The lines connecting two entities (rectangle/triangle/circle) indicate that vertices corresponding to one entity is adjacent to the vertices representing the other entity.

are from outside G'. Therefore, $G' \oplus S'$ has an induced $T_{x-1,y-1}$, which contains an induced $K_{1,y}$, a contradiction. Let one of the centers, say u, be from $V(G')$, and the other, say u', is from W_u. Then $u' \in Y_{u_{x+1}}$. Since the size of the maximum independent set in the neighborhood of any vertex in $Y_{u_{x+1}}$ in W_u is x, we obtain that there is a $K_{1,y}$ induced in $G' \oplus S'$, which is a contradiction. Assume that both a and b are from the new vertices created in G. Since W_u and $W_{u'}$ are not adjacent for two vertices $u, u' \in V(G')$, we obtain that both a and b are from W_u for some vertex $u \in V(G')$. Let one of the centers, say v of $T_{x,y}$ is from $Y_{u_{x+1}}$. Then the other center, say v' is from any of the sets Y_{u_j} for $1 \leq j \leq x$. We observe that for every vertex $w \in Y_{u_j}$, the size of the maximum independent set in the neighborhood of w in $W_u \setminus Y_{u_{x+1}}$ is $2 < y$ (recall that each $U_{j\ell}$ induces a $\overline{T}$ which is $3K_1$-free). Therefore, $v' = a$, the x-center of $T_{x,y}$, and v is the y-center of the $T_{x,y}$. Further, $x \leq 2$. But, the size of the maximum independent set in the neighborhood of v, excluding the clique containing v', is x. This implies that $x = y \leq 2$, which is a contradiction. Therefore, both a and b are from $W_u \setminus Y_{u_{x+1}}$. It is straight-forward to verify that there are no two adjacent vertices a, b in $W_u \setminus Y_{u_{x+1}}$, and an independent set I of size $x + y$ in $W_u \setminus \{a, b\}$ such that a is adjacent to and b is non-adjacent to x vertices in I, and b is adjacent to and a is non-adjacent to y vertices in I. $\square$

The converse of the lemma turns out to be true as well.

Lemma 4.16. *If $G \oplus S \in \mathcal{F}(T_{x,y})$ for some $S \subseteq V(G)$, then $G' \oplus S' \in \mathcal{F}(K_{1,y})$, where $S' = S \cap V(G)$.*

Proof. We observe that for every vertex u, set U_{ij} induces a $\overline{T_{x,y}}$. Therefore, S cannot contain all the vertices in U_{ij}. If Y_{u_i}, for $1 \leq i \leq x$, is a subset of S, then Y_{u_i} and one vertex each from $U_{ij} \setminus S$ (for $1 \leq j \leq x + 2$) induce a $T_{x,y}$ in $G \oplus S$. Therefore, at least one vertex of Y_{u_i} is not in S. If $Y_{u_{x+1}}$ is a subset of S, then $Y_{u_{x+1}}$ and one vertex from $Y_{u_x} \setminus S$, and one vertex each from $U_{(x+1)j} \setminus S$ (for $1 \leq j \leq x + 1$) induce a $T_{x,y}$. Therefore, at least one vertex of $Y_{u_{x+1}}$ is not in S. Assume that there is a $K_{1,y}$ induced by a set A in $G' \oplus S'$. Then, A along with one vertex each from $Y_{u_j} \setminus S$, for $1 \leq j \leq x + 1$, induce a $T_{x,y}$ in $G \oplus S$, which is a contradiction. $\square$

Now, Lemma 4.14 follows from Lemma 4.15 and Lemma 4.16. Further, Theorem 4.13 follows from Lemma 4.14 and Proposition 2.3.

4.4 Tristar graphs

Recall that, in Section 4.2, as a corollary of the main result we have resolved some cases of tristar graphs: we proved that SC-TO-$\mathcal{F}(T_{x,y,z})$ is hard if $x \geq 2$ or if $x = 1, y = 0, z \geq 3$. In this section, we handle the rest of the cases when $x = 1$ and $y \geq 1$, except for a finite number of cases. First we give a linear reduction from SC-TO-$\mathcal{F}(T_{y,z-1})$ to SC-TO-$\mathcal{F}(T_{x,y,z})$. This will take care of the cases when $y \geq 5$ or $z \geq 6$ (recall that $T_{y,z-1}$ is hard if $y \geq 5$ or $z \geq 6$). But, for the reduction to work, there is an additional constraint that $z \geq 3$. So, to handle the case when $z \leq 3$, we introduce another reduction which is from SC-TO-$\mathcal{F}(K_{1,y})$ and does not have any constraint on z. Thus, the main result of this section is the following.

Theorem 4.17. *Let $1 \leq x \leq z$, and $y \geq 0$ be integers such that $y \geq 5$ or $z \geq 6$. Then* SC-TO-*$\mathcal{F}(T_{x,y,z})$ is NP-Complete. Further, the problem cannot be solved in time $2^{o(|V(G)|)}$, unless the ETH fails.*

First we introduce the reduction from SC-TO-$\mathcal{F}(T_{y,z-1})$ to SC-TO-$\mathcal{F}(T_{x,y,z})$.

Lemma 4.18. *Let x, y, z be integers such that $x = 1, y \geq 1$, and $z \geq 3$. Then there is a linear reduction from* SC-TO-*$\mathcal{F}(T_{y,z-1})$ to* SC-TO-*$\mathcal{F}(T_{x,y,z})$.*

Let T be a tristar graph $T_{x,y,z}$ satisfying the properties stated in Lemma 4.18. Construction 6 is used for the reduction from SC-TO-$\mathcal{F}(T_{y,z-1})$ to SC-TO-$\mathcal{F}(T_{x,y,z})$. The reduction is similar to that used for bistars, but simpler.

Construction 6. *Let (G', y, z) be the input to the construction, where G' is a graph, $y \geq 1$, and $z \geq 3$ are integers. Let $t = y + z + 4$. For every vertex u of G', introduce a K_z, denoted by K_u, in the neighbourhood of u. Further, introduce $t - z$ copies of $\overline{T}$ denoted by X_{u_i}, for $1 \leq i \leq t - z$, in the neighbourhood of K_u. The union of X_{u_i}, for $1 \leq i \leq x$, is denoted by X_u. The adjacency among these X_{u_i}s and K_u is in such a way that taking z vertices from the complement of K_u and one vertex each from X_{u_i}s induces a T. Let W_u denote the set of all vertices introduced for a vertex u in G', i.e., $W_u = K_u \cup X_u$. We observe that W_u and W_v are non-adjacent for any two vertices $u, v \in V(G')$. This completes the construction of the graph G (see Figure 12 for an example).*

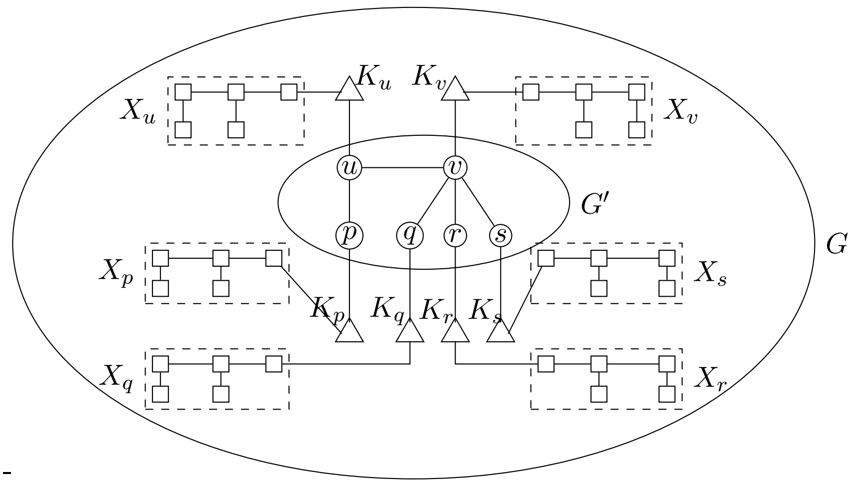


Figure 12: An example of Construction 6 with $x = 1$ and $y = 1$ and $z = 4$. Each bold rectangle represents a $\overline{T_{x,y,z}}$ and each triangle represents a K_4. The lines connecting two entities (rectangle/triangle/circle) represents all possible edges between the vertices in those entities.

Lemma 4.19. *If $G' \oplus S' \in \mathcal{F}(T_{y,z-1})$ for some $S' \subseteq V(G')$, then $G \oplus S' \in \mathcal{F}(T_{x,y,z})$.*

Proof. Let a $T_{x,y,z}$ be induced by a set A in $G \oplus S'$. Let a, b, c be the x-center, y-center, and the z-center, respectively of the $T_{x,y,z}$ induced by A. Assume that $a, b, c \in V(G')$. Since a, b, and c get only at most one leaf from W_a, W_b, and W_c respectively, there is a $T_{y,z-1}$ in $G' \oplus S'$. Assume that $a, b \in V(G')$ and c is from W_b. Then, c can have at most two leaves from X_b, which is a contradiction. The case when $b, c \in V(G')$ and $a \in W_b$ gives a contradiction as there is a $T_{y,z-1}$ in $G' \oplus S'$. Assume that only a is from G' and b, c are from W_a. Then b does not get any leaf, which means that $y = 0$, which is a contradiction. The case when c is from G' and a, b are from W_a can be handled in a similar way. Note that it is not possible that only b, among the centers, is from G', as the neighborhood of b in W_b is a clique. Assume that $a, b, c \in W_u$ for some vertex u in G'. Then A must be subset of $\{u\} \cup K_u \cup X_u$ for some $u \in G'$. It is straight-forward to verify that there is no induced $T_{x,y,z}$ in the graph induced by $\{u\} \cup K_u \cup X_u$ in $G \oplus S'$. This completes the proof. $\square$

The converse of the lemma turns out to be true as well.

Lemma 4.20. *If $G \oplus S \in \mathcal{F}(T_{x,y,z})$ for some $S \subseteq V(G)$, then $G' \oplus S' \in \mathcal{F}(T_{y,z-1})$, where $S' = S \cap V(G')$.*

Proof. Since each X_{u_i} induces a $\overline{T_{x,y,z}}$, at least one of its vertices is not in S. Therefore, at least one vertex of K_u is not in S, otherwise, the complement of K_u along with one vertex each from $X_{u_i} \setminus S$ induces a $T_{x,y,z}$. Then, if $G' \oplus S'$ contains a $T_{y,z-1}$ induced by a set A (say), then there is a $T_{x,y,z}$ in $G \oplus S$ induced by A along with one vertex each from $X_a \setminus S, X_b \setminus S$, and $X_c \setminus S$, where b and c are the y-center and $(z-1)$- center respectively of $T_{y,z-1}$ and a is one of the leaf of b in $T_{y,z-1}$. This completes the proof. □

Lemma 4.18 follows from Lemma 4.19 and Lemma 4.20. Now, we introduce the reduction from SC-TO-$\mathcal{F}(K_{1,y})$ to SC-TO-$\mathcal{F}(T_{x,y,z})$.

Lemma 4.21. *Let x, y, z be integers such that $1 \leq x \leq z, y \geq 3$. Then there is a linear reduction from* SC-TO-$\mathcal{F}(K_{1,y})$ *to* SC-TO-$\mathcal{F}(T_{x,y,z})$.

Let $T_{x,y,z}$ be a tristar graph satisfying the properties stated in Lemma 4.21. Construction 7 is used for the reduction.

Construction 7. *Let (G', x, y, z) be the input to the construction, where G' is a graph, and $z \geq x \geq 1$ and $y \geq 3$, are integers. Let $t = x + y + z + 3$. For every vertex u of G', introduce two K_ys, denoted by P_{u1} and P_{u2}, in the neighbourhood of u. Further, introduce x copies of $\overline{T_{x,y,z}}$ denoted by X_{u_i}, for $1 \leq i \leq x$, in the neighbourhood of P_{u1}, and z copies of $\overline{T_{x,y,z}}$ denoted by Z_{u_i}, for $1 \leq i \leq z$, in the neighbourhood of P_{u2}. The union of X_{u_i}, for $1 \leq i \leq x$, is denoted by X_u, and the union of Z_{u_i}s, for $1 \leq i \leq z$ is denoted by Z_u. Introduce a set X'_u which contains $t - y$ copies of $\overline{T_{x,y,z}}$s, denoted by X'_{u_i} for $1 \leq i \leq t - y$. The adjacency among these X'_{u_i}s is in such a way that taking y vertices from the complement of P_{u1} and one vertex each from X'_{u_i}s induces a $T_{x,y,z}$. Similarly, introduce a set Z'_u which contains $t - y$ copies of $\overline{T_{x,y,z}}$, denoted by Z'_{u_i}, for $1 \leq i \leq t - y$. The adjacency among these Z'_{u_i}s is in such a way that taking y vertices from the complement of P_{u2} and one vertex each from Z'_{u_i}s induces a $T_{x,y,z}$. Further, X_u is adjacent to X'_u and Z_u is adjacent to Z'_u. Let W_u denote the set of all vertices introduced for a vertex u in G', i.e., $W_u = P_{u1} \cup P_{u2} \cup X_u \cup Z_u \cup X'_u \cup Z'_u$. This completes the construction of the graph G (see Figure 13 for an example).*

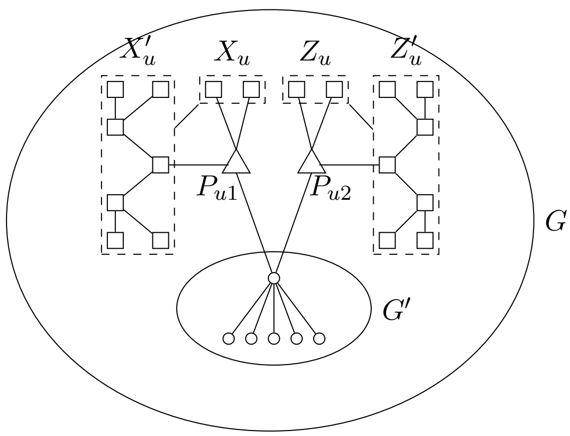


Figure 13: An example of Construction 7 with $x = 2$, $y = 5$, and $z = 2$. Each bold rectangle represents a $\overline{T_{x,y,z}}$ and each triangle represents a K_5. The lines connecting two entities (rectangle/triangle/circle) indicate the existence of all possible edges between the vertices of the entities.

It is straight-forward to verify that the following observation holds true due to the adjacency between X_u and X'_u, and between Z_u and Z'_u.

Observation 4.22. *Let u be any vertex in G'. Then there is no P_4 induced by vertices in W_u such that at least one of the internal vertex of the P_4 is from either P_{u1} or P_{u2}.*

Lemma 4.23. *If $G' \oplus S' \in \mathcal{F}(K_{1,y})$ for some $S' \subseteq V(G')$, then $G \oplus S' \in \mathcal{F}(T_{x,y,z})$.*

Proof. Let a $T_{x,y,z}$ be induced by a set A in $G \oplus S'$. Let a, b, c be the x-center, the y-center, and the z-center, respectively of the $T_{x,y,z}$ induced by A. Assume that $a, b, c \in V(G')$. Since b gets only at most two leaves from W_b, there is a $K_{1,y}$ in $G' \oplus S'$ induced by a, b, c, and $y-2$ leaves of b from the $T_{x,y,z}$. Assume that $a, b \in V(G')$ and c is from W_b. Then b can get at most one leaf from W_b. Therefore, there is a $K_{1,y}$ in $G' \oplus S'$ induced by a, b, and $y-1$ leaves of b from the $T_{x,y,z}$. The case when $b, c \in V(G')$ and $a \in W_b$ is symmetrical. Assume that only b is from G' and a and c are from W_b. Then none of the leaves of b are from W_b. Therefore, there is a $K_{1,y}$ induced by b and its y leaves from the $T_{x,y,z}$. Assume that $a \in V(G')$ and b and c are from W_a. Then b is from P_{a1} or P_{a2}. Then there is a P_4 (due to the fact that $y, z \geq 1$) induced by some vertices in W_a such that one of the internal vertex of the P_4 is from either P_{a1} or P_{a2}. By Observation 4.22, this is not true. This is a contradiction. The case when $c \in V(G')$ and a and b are from W_c is symmetrical. Therefore, $a, b, c \in W_u$ for some $u \in V(G')$. If b is from P_{u1} or P_{u2}, then we get a contradiction using Observation 4.22. Therefore, b is from $W_u \setminus (P_{u1} \cup P_{u2})$. In this case, it can be verified that, since $y \geq 3$, there is no $T_{x,y,z}$ where at most one leaf is from G'. ☐

The converse of the lemma also is true.

Lemma 4.24. *If $G \oplus S \in \mathcal{F}(T_{x,y,z})$ for some $S \subseteq V(G)$, then $G' \oplus S' \in \mathcal{F}(K_{1,y})$, where $S' = S \cap V(G')$.*

Proof. Since each X_{u_i} induces a $\overline{T_{x,y,z}}$, at least one of its vertices is not in S. The case is same with X'_{u_i}s, Z_{u_i}s, and Z'_{u_i}s. Therefore, at least one vertex of P_{u1} is not in S, otherwise, the complement of P_{u1} along with one vertex each from $X'_{u_i} \setminus S$ induces a $T_{x,y,z}$. Similarly, P_{u2} is not a subset of S. Then, if $G' \oplus S'$ contains a $K_{1,y}$ induced by a set A (say), then there is a $T_{x,y,z}$ in $G \oplus S$ induced by A along with one vertex each from $P_{u1} \setminus S, P_{u2} \setminus S$, and one vertex each from $X_{u_i} \setminus S$, for $1 \leq i \leq x$, and one vertex each from $Z_{u_i} \setminus S$, for $1 \leq i \leq z$. This completes the proof. ☐

Lemma 4.21 follows from Lemma 4.23 and Lemma 4.24. Now, we are ready to prove Theorem 4.17.

Proof of Theorem 4.17. Let the integers x, y, z satisfy the constraints given in the theorem, i.e., $1 \leq x \leq z$, $y \geq 0$, and either $y \geq 5$ or $z \geq 6$. If $x \geq 2$ or if $y = 0$, then the statements follow from Corollary 4.12. Assume that $x = 1$ and $y \geq 1$. Let $z \geq 6$. Then by Lemma 4.18, there is a linear reduction from SC-TO-$\mathcal{F}(T_{y,z-1})$ to SC-TO-$\mathcal{F}(T_{x,y,z})$. Then the statements follow from Theorem 4.13. Let $y \geq 5$. Then by Lemma 4.21, there is a linear reduction from SC-TO-$\mathcal{F}(K_{1,y})$ to SC-TO-$\mathcal{F}(T_{x,y,z})$. Then the statement follows from Proposition 2.3. ☐

4.5 Paths

By Proposition 2.2, SC-TO-$\mathcal{F}(P_\ell)$ is hard for every $\ell \geq 7$. Here, we extend the result to P_6.

Theorem 4.25. SC-TO-$\mathcal{F}(P_6)$ *is NP-Complete. Further, the problem cannot be solved in time $2^{o(|V(G)|)}$, unless the ETH fails.*

Proposition 2.2 and Theorem 4.25 imply Corollary 4.26.

Corollary 4.26. *Let $\ell \geq 6$ be an integer. Then* SC-TO-$\mathcal{F}(P_\ell)$ *is NP-Complete. Further, the problem cannot be solved in time $2^{o(|V(G)|)}$, unless the ETH fails.*

Construction 8 is used for SC-TO-$\mathcal{F}(P_6)$. The reduction is from 3-SAT. The reduction is similar to other reductions that we introduced from various boolean satisfiability problems. Since a P_6 has neither a $4K_1$ nor a K_4, the usual technique of keeping an independent set of size 4 of the literal vertices does not work. To overcome this hurdle, we introduce a vertex in the clause gadgets.

Construction 8. *Let Φ be a* 3-SAT *formula with n variables $X_1, X_2, \cdots, X_n$, and m clauses $C_1, C_2, \cdots, C_m$. We construct the graph G_Φ as follows.*

- *For each variable X_i in Φ, the variable gadget, also named X_i, consists of two special sets $X_{i1} = \{x_i\}, X_{i2} = \{\overline{x_i}\}$, and four other sets $X_{i3}, X_{i4}, X_{i5}, X_{i6}$, where each of the set in $\{X_{i3}, X_{i4}, X_{i5}, X_{i6}\}$ induces a $\overline{P_6}$. The set X_{i1} is adjacent to X_{i3} which is adjacent to X_{i5}. Similarly, the set X_{i2} is adjacent to X_{i4} which is adjacent to X_{i6}. Let $X = \bigcup_{i=1}^{i=n} X_i$. The vertices in $X_{i1}(= \{x_i\})$ and $X_{i2}(= \{\overline{x_i}\})$ (for $1 \leq i \leq n-1$) are called literal vertices, and let L be the union of all literal vertices. The set L induces an independent set of size $2n$.*

- *For each clause C_i of the form $(\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$ in Φ, the clause gadget, also named as C_i, consists of a set C_{i2} which contains a single vertex c_{i2} and two copies of $\overline{P_6}$ denoted by C_{i1} and C_{i3}. The sets C_{i1} and C_{i2} are adjacent. Let the three sets introduced (in the previous step) for the literals $\ell_{i1}, \ell_{i2}, \ell_{i3}$ be denoted by $L_i = \{y_{i1}, y_{i2}, y_{i3}\}$. Each C_{ij} is adjacent to y_{ij}, for $1 \leq j \leq 3$, and C_{i3} is adjacent to y_{i2}. In addition to this, every vertex in C_i is adjacent to all literal vertices corresponding to literals not in C_i with an exception- C_{i2} is adjacent to none of the literal vertices corresponding to literals not in C_i. The union of all clause gadgets C_i is denoted by C, and their vertices are called clause vertices.*

- *For all $i \neq j$, make the set C_i adjacent to the set C_j, and then remove the edge between C_{i2} and C_{j2}.*

- *For $1 \leq i \leq n$, the vertices in $X_i \setminus \{x_i, \overline{x_i}\}$ are adjacent to $V(G) \setminus X_i$.*

  This completes the construction of the graph G_Φ (see Figure 14 for an example).

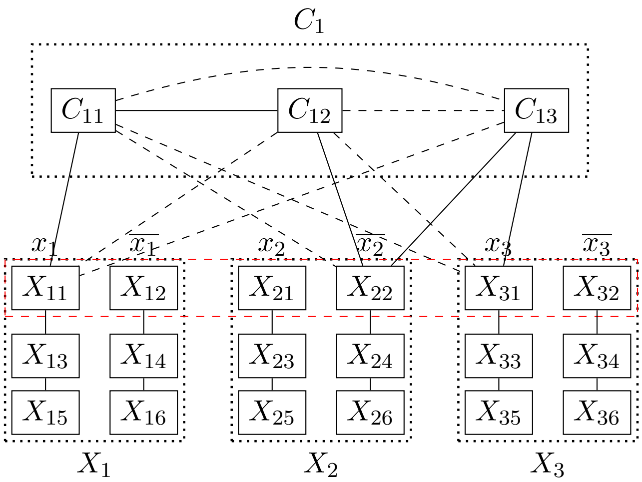


Figure 14: An example of Construction 8 for the formula $\Phi = (x_1 \vee \overline{x_2} \vee x_3)$. The bold lines (respectively dashed lines) connecting two rectangles indicate that each vertex in one rectangle is adjacent (respectively non-adjacent) to all vertices in the other rectangle. If there is no line shown between two rectangles, then the vertices in them are adjacent, with the exceptions – (i) all the vertices in the red rectangle (dashed) together form an independent set; (ii) if there is no line shown between two rectangles in the dotted rectangles, then the rectangles in them are non-adjacent.

To prove the forward direction of the correctness of the reduction, we need a few lemmas to handle some special cases arising in the forward direction.

Lemma 4.27. *Let Φ be a yes-instance of* 3-SAT *and ψ be a truth assignment satisfying Φ. Then there exists no set A of vertices such that $A \subseteq (L \cap C)$, $(A \cap C) \subseteq (C_i \cup C_j)$, $|A \cap C_i| = 2$, $|A \cap C_j| = 1$, and A induces a P_6 in $G_\Phi \oplus S$, where S is the union of the clause vertices c_{i2}, for $1 \leq i \leq m$, and the set of literal vertices whose corresponding literals were assigned TRUE by ψ.*

Proof. Let $A \cap C_i = \{c_{ia}, c_{ib}\}$ and $A \cap C_j = \{c_j\}$. Clearly, $c_{ia}c_jc_{ib}$ is a P_3. Then $A \cap L$ induces either a P_3 or a $K_2 + K_1$. The former is a contradiction as L induces $K_n + nK_1$. Assume that $A \cap L$ induces a $K_2 + K_1$. Let $A \cap L = \{q_1, q_2, q_3\}$ and let q_2q_3 be the edge in the $K_2 + K_1$ induced by $A \cap L$. Note that c_j is adjacent to none of the vertices in $\{q_1, q_2, q_3\}$ Since there is no vertex in $C_{\ell 1} \cup C_{\ell 3}$ (for any $1 \leq \ell \leq m$) which is non-adjacent to three vertices in L, we obtain that $c_j \in C_{j2}$. Therefore, $c_j \in S$. Since c_j is in S, c_j is non-adjacent to at most one vertex in S– recall that C_{j2} is non-adjacent to all vertices in L except y_{j2}, thereby adjacent to all vertices in S except y_{j2} if it is in S. Thus, c_j is adjacent to at least one of q_2, q_3, which leads to a contradiction. □

Lemma 4.28. *Let Φ be a yes-instance of* 3-SAT *and ψ be a truth assignment satisfying Φ. Then there exists no set A of vertices such that $A \subseteq (L \cap C)$, $A \cap C \subseteq C_i$ and A induces a P_6 in $G_\Phi \oplus S$, where S is the union of the clause vertices c_{i2}, for $1 \leq i \leq m$, and the set of literal vertices whose corresponding literals were assigned TRUE by ψ.*

Proof. If $|A \cap C_i| = 1$, then $A \cap L$ induces a graph which has an induced P_3, which is a contradiction. If $|A \cap C_i| = 2$, then $A \cap C_i$ induces either a K_2 or a $2K_1$. Assume that $A \cap C_i$ induces a K_2. Then, $A \cap L$ induces either a P_4, or a $P_3 + K_1$, or a $2K_2$ - all of them lead to contradictions. Assume that $A \cap C_i$ induces a $2K_1$. Then, $A \cap L$ induces either a P_4, or a $P_3 + K_1$, or a $2K_2$, or a $K_2 + 2K_1$ - the first three cases lead to contradictions.

Now assume that $A \cap L$ induces a $K_2 + 2K_1$. Let c_{ia} and c_{ib} be the vertices from $A \cap C$. Let q_1, q_2, q_3, q_4 be the vertices from $A \cap L$ such that q_3q_4 induces an edge. Thus, $\{q_1, q_2\} \notin S$ and $\{q_3, q_4\} \in S$– recall that the vertices in $S \cap L$ induces a clique. Without loss of generality, assume that c_{ia} is adjacent to q_1 and q_2 and c_{ib} is adjacent to q_2 and q_3, i.e., $q_1c_{ia}q_2c_{ib}q_3q_4$ is the P_6 induced by A in $G_\Phi \oplus S$. Since C_{i1} and C_{i2} are adjacent, $A \cap C_i$ contains vertices from either C_{i1} and C_{i3}, or C_{i2} and C_{i3}. Both c_{ia} and c_{ib} are non-adjacent to two vertices in $\{q_1, q_2, q_3, q_4\}$. But none of the vertices in $C_{i1} \cup C_{i3}$ is non-adjacent to two vertices in $G_\Phi \oplus S$. This implies that $c_{ia}, c_{ib} \notin C_{i1} \cup C_{i3}$, which is a contradiction.

If $|A \cap C_i| = 3$, then $A \cap C_i$ induces a $K_2 + K_1$. Clearly, $c_{i1} \in C_{i1}$, $c_{i2} \in C_{i2}, c_{i3} \in C_{i3}$ are the three vertices in $C_i \cap L$, where $c_{i1}c_{i2}$ is the edge in the $K_2 + K_1$ induced by $A \cap C_i$. Then $A \cap L$ induces either a P_3 or a $K_2 + K_1$ or a $3K_1$. The first case gives a contradiction. Assume that $A \cap L$ induces a $K_2 + K_1$. Let q_1, q_2, q_3 be the vertices from $A \cap L$ in which q_1q_2 is the edge. Let $p_1p_2p_3p_4p_5p_6$ be the P_6 induced by A. Now there can be four cases- (i) p_1, p_2, p_4 belong to $A \cap C_i$ and p_3, p_5, p_6 belong to $A \cap L$; (ii) p_2, p_3, p_6 belong to $A \cap C_i$ and p_1, p_4, p_5 belong to $A \cap L$; (iii) p_3, p_4, p_6 belong to $A \cap C_i$, p_1, p_2, p_5 belong to $A \cap L$; (iv) p_1, p_2, p_5 belong to $A \cap C_i$, p_3, p_4, p_6 belong to $A \cap L$.

Now consider the case (i) : i.e., p_1, p_2, p_4 belong to $A \cap C_i$ and p_3, p_5, p_6 belong to $A \cap L$. Clearly, p_4 is c_{i3} and q_2 and q_3 are adjacent to c_{i3}. Further, either c_{i1} or c_{i2} is adjacent to q_3. and both c_{i1} and c_{i2} are non-adjacent to both q_1 and q_2. But $\{q_1, q_2, c_{i2}\} \in S$. This is a contradiction as then c_{i2} must be adjacent to either q_1 or q_2.

Now consider the case (ii) and (iii): In both cases, p_6 must be c_{i3} and hence c_{i3} is non-adjacent to at least two vertices among $\{q_1, q_2, q_3\}$. This is a contradiction, as c_{i3} is non-adjacent to only one vertex (y_{i1}) in $G_\Phi \oplus S$.

Now consider case (iv): p_1, p_2, p_5 are in $A \cap C_i$. Then $c_{i3} = p_5$ and either $c_{i1} = p_1$ or $c_{i2} = p_1$. If $c_{i2} = p_1$, then c_{i2} is not adjacent to $\{q_1, q_2\} \in S$, which is a contradiction as c_{i2} is non-adjacent to only at most one vertex (y_{i2}) in $S \cap L$. Therefore, $c_{i1} = p_1$. Then c_{i1} is non-adjacent to all the three vertices q_1, q_2, and q_3, which is a contradiction.

Assume that $A \cap L$ induces a $3K_1$. Let q_1, q_2, q_3 be the vertices from $A \cap L$. Let $p_1p_2p_3p_4p_5p_6$ be the P_6 induced by A. This means that $\{p_2, p_3, p_5\} \in A \cap C_i$, where p_2p_3 is an edge. Clearly, c_{i3} is p_5. There are two cases - either c_{i1} is p_2 or c_{i2} is p_2. Assume that c_{i1} is p_2. Then, without loss of generality, assume that $q_1c_{i1}c_{i2}q_2c_{i3}q_3$ is the P_6. This means that p_2 (c_{i1}) is non-adjacent to q_2, q_3, which implies that $\{q_2, q_3\} = \{y_{i2}, y_{i3}\}$. Similarly, p_5 (c_{i3}) is non-adjacent to q_1, which implies that $q_1 = y_{i1}$. Thus, $A \cap L \subseteq L_i$. Since c_{i2} and at least one vertex in L_i belong to S, A induces a graph which is not isomorphic to P_6, which is a contradiction. Now assume that $p_3 = c_{i1}$, $p_2 = c_{i2}$, $p_5 = c_{i3}$, $p_1 = q_1$, $p_4 = q_2$, and $p_6 = q_3$. This means that $p_3 = c_{i1}$ is non-adjacent to q_1, q_3, which implies that $\{q_1, q_3\} = \{y_{i2}, y_{i3}\}$. Similarly, P_5 (c_{i3}) is non-adjacent to q_1, which implies that $q_1 = y_{i1}$. This gives a contradiction as q_1 can not be corresponding two different vertices. □

Now, we prove the forward direction with the help of Lemma 4.27 and Lemma 4.28.

Lemma 4.29. *Let Φ be a yes-instance of* 3-SAT *and ψ be a truth assignment satisfying Φ. Then $G_\Phi \oplus S \in \mathcal{F}(P_6)$ where S is the union of the clause vertices c_{i2}, for $1 \leq i \leq m$ and the set of literal vertices whose corresponding literals were assigned TRUE by ψ.*

Proof. Let $G_\Phi \oplus S$ contain a P_6 induced by A (say). We prove the lemma with the help of a set of claims.

Claim 1: A is not a subset of X_i, for $1 \leq i \leq n$.

Assume that A is a subset of X_i, for some $1 \leq i \leq n$. Since P_6 is a prime graph and $\overline{P_6}$ is not isomorphic to P_6, $A \cap Y \leq 1$, where Y is a module isomorphic to $\overline{P_6}$. Thus, $|A \cap X_{ij}|$ is at most one. Since X_i has six sets and each of them contains at most one vertex of A, then each X_{ij} (for $1 \leq j \leq 6$) has exactly one vertex of A. Recall that $\{x_i, \overline{x_i}\}$ is not a subset of S. Hence, we obtain that the graph induced by A is a $2P_3$ which is not isomorphic to P_6, which is a contradiction.

Claim 2: For $1 \leq i \leq n$, let $X'_i = X_i \setminus \{x_i, \overline{x_i}\}$ and $\overline{X_i} = V(G_\Phi) \setminus X_i$. If $|A \cap X'_i| \geq 1$, then $A \cap \overline{X_i} = \emptyset$. Similarly, if $|A \cap \overline{X_i}| \geq 1$, then $A \cap X'_i = \emptyset$.

For a contradiction, assume that A contains at least one vertex from X'_i and at least one vertex from $\overline{X_i}$. Since X'_i and $\overline{X_i}$ are adjacent, either $|A \cap X'_i| = 1$ or $|A \cap \overline{X_i}| = 1$.

Assume that $A \cap X'_i = \{u\}$. We note that $V(G_\Phi) \setminus (X'_i \cup \overline{X_i}) = \{x_i, \overline{x_i}\}$. Therefore, $|A \cap \overline{X_i}| \geq 3$. This implies that there is a claw ($K_{1,3}$) formed by u and three vertices in $\overline{X_i}$, which is a contradiction as there is no claw in a P_6. Assume that $A \cap \overline{X_i} = \{u\}$. Then with the same argument as given above, we obtain that the graph induced by A contains a claw as a subgraph, which is a contradiction.

Claim 3: A is not a subset of L, the set of all literal vertices.

This follows from the fact that L induces a $K_n + (n)K_1$ in $G_\Phi \oplus S$.

Claim 4: A cannot have nonempty intersections with three distinct clause gadgets C_i, C_j, and C_ℓ.

Claim 5: There exists no C_i and C_j ($i \neq j$) such that $|A \cap C_i| \geq 2$ and $|A \cap C_j| \geq 2$.

Claim 4 and 5 follow from the fact that C_i is adjacent to C_j (for $i \neq j$), in $G_\Phi \oplus S$ and the fact that there is neither a triangle nor a C_4 in a P_6.

Claim 6: A is not a subset of C.

Since P_6 is a prime graph and $\overline{P_6}$ is not isomorphic to P_6, $A \cap C_{ij}$ has at most one vertex. Therefore, $A \cap C_i$ has at most three vertices. By Claim 4, A cannot have nonempty intersections with three clause gadgets C_i, C_j and C_ℓ. Therefore, A has nonempty intersection with exactly two sets C_i and C_j and $|A \cap C_i| = |A \cap C_j| = 3$, which is a contradiction by Claim 5.

Claim 7: If $|A \cap C_i| > 2$, then $A \cap C_j = \emptyset$ ($i \neq j$).

If A contains three vertices from C_i and at least one vertex from C_j, then there is a claw in the graph induced by A as C_i and C_j are adjacent in $G_\Phi \oplus S$.

Now, we are ready to prove the lemma. By Claim 1, A is not a subset of X_i (for $1 \leq i \leq n$). By Claim 2, A cannot have vertices from both $X_i \setminus \{x_i, \overline{x_i}\}$ and $\overline{X_i}$ (for $1 \leq i \leq n-1$). This implies that $A \subseteq L \cup C$. By Claim 3, A cannot be a subset of L and by Claim 6, A cannot be a subset of C. Therefore, A contains vertices from both L and C. By Claim 4, A cannot have nonempty intersections with three distinct sets C_i, C_j and C_ℓ. Therefore, $A \cap C \subseteq (C_i \cup C_j)$. Assume that A has nonempty intersection with both C_i and C_j. By Claim 5 and Claim 7, we can assume that $|A \cap C_j| = 1$ and $|A \cap C_i| \leq 2$. Assume that $|A \cap C_i| = 2$. Then the statement follows from Lemma 4.27. Now, assume that $A \cap C_i = \{c_i\}$ and $A \cap C_j = \{c_j\}$. This means that $A \cap L$ induces either a P_4, or a $P_3 + K_1$, or a $2K_2$ - all of them lead to contradictions. Thus, we can conclude that $A \cap C$ contains vertices from C_i only. Then the statement follows from Lemma 4.28. This completes the proof. □

The converse is also true.

Lemma 4.30. *Let Φ be an instance of* 3-SAT. *If $G_\Phi \oplus S \in \mathcal{F}(P_6)$ for some $S \subseteq V(G_\Phi)$ then there exists a truth assignment satisfying Φ, i.e., TRUE to at least one literal per clause.*

Proof. Let $G_\Phi \oplus S \in \mathcal{F}(P_6)$ for some $S \subseteq V(G_\Phi)$. We want to find a satisfying truth assignment of Φ. We know that each of the sets C_{i1} and C_{i3}, for $1 \leq i \leq m$, induces a $\overline{P_6}$. Therefore, each such set has at least one vertex not in S. Hence at least one vertex in $L_i = \{y_{i1}, y_{i2}, y_{i3}\}$ must belong to S, otherwise there is an induced P_6 in $G_\Phi \oplus S$ by vertices in L_i and one vertex each from C_{i2}, $C_{i1} \setminus S$, and $C_{i3} \setminus S$.

Similarly, each set X_{ij}, for $1 \leq i \leq n$ and $3 \leq j \leq 6$, induces a $\overline{P_6}$. Therefore, X_{ij} has at least one vertex untouched by S. Hence, if both x_i and $\overline{x_i}$ are in S, then there is an induced copy of P_6 in $G_\Phi \oplus S$, which is a contradiction. Therefore, $\{x_i, \overline{x_i}\}$ is not a subset of S. Now, it is straight-forward to verify that assigning TRUE to each literal corresponding to the literal sets which are subsets of S is a satisfying truth assignment for Φ. $\square$

Now, Theorem 4.25 follows from Lemma 4.29 and Lemma 4.30.

4.6 Subdivisions of claw

A subdivision of a claw has exactly three leaves. Due to this, we cannot handle them using the reduction used to handle trees with 4 leaves (Theorem 4.5). Let $T = C_{x,y,z}$ be a subdivision of claw, where $x \leq y \leq z$. If $x = y = 1$, then T is obtained from P_{z+2} by duplicating a leaf. Therefore, we can use Lemma 3.1 and Corollary 4.26 to prove the hardness, when $z \geq 4$. If $y > 1$, then T is prime and if T has at least 9 vertices, then $\overline{T}$ is 5-connected (we will prove this in this section). Then the hardness results for 5-connected prime graphs can be used to prove the hardness for $\overline{T}$ and hence for T (Proposition 2.4). But, there is a special subdivision of claw, $C_{1,2,4}$, which is not handled by any of these reductions. Further, there is an infinite family of trees, which is obtained by duplicating the leaf adjacent to the center of the claw in $C_{1,2,4}$, not handled by Theorem 4.5. This requires us to handle $C_{1,2,4}$ separately. We will start this section with a reduction for $C_{1,2,4}$ and end by proving the hardness of SC-TO-$\mathcal{F}(T)$ if T is not among 7 specific subdivisions of claw.

Let T be the subdivided claw $C_{1,2,4}$ as shown in Figure 15.

Theorem 4.31. SC-TO-$\mathcal{F}(C_{1,2,4})$ *is NP-Complete. Further, the problem cannot be solved in time* $2^{o(|V(G)|)}$*, unless the ETH fails.*

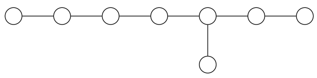


Figure 15: The subdivided claw $C_{1,2,4}$

Construction 9 is used for a reduction from 4-SAT$_{\geq 2}$ to SC-TO-$\mathcal{F}(C_{1,2,4})$. The reduction is similar to the one used to prove hardness for trees having 3 internal vertices and 4 leaves.

Construction 9. *Let Φ be a 4-SAT$_{\geq 2}$ formula with n variables $X_1, X_2, \cdots, X_n$, and m clauses $C_1, C_2, \cdots, C_m$. We construct the graph G_Φ as follows.*

- *For each variable X_i in Φ, the variable gadget, also named X_i, consists of two special sets $X_{i1} = \{x_i\}, X_{i2} = \{\overline{x_i}\}$, and six other sets $X_{i3}, X_{i4}, X_{i5}, X_{i6}, X_{i7}, X_{i8}$, where each of the set in $\{X_{i3}, X_{i4}, X_{i5}, X_{i6}, X_{i7}, X_{i8}\}$ induces a $\overline{C_{1,2,4}}$. The set X_{ij} is adjacent to $X_{i(j+2)}$, for $1 \leq j \leq 5$. Further, the sets X_{i2} and X_{i8} are adjacent. Let $X = \bigcup_{i=1}^{i=n} X_i$. The vertices x_i and $\overline{x_i}$ are called literal vertices, and L is the set of all literal vertices. The set L forms an independent set of size $2n$.*

- *For each clause C_i of the form $(\ell_{i1} \vee \ell_{i2} \vee \ell_{i3} \vee \ell_{i4})$ in Φ, the clause gadget also named as C_i consists of four copies of $\overline{C_{1,2,4}}$s denoted by C_{i1}, C_{i12}, C_{i3}, and C_{i4}. Let the four vertices introduced (in the previous step) for the literals $\ell_{i1}, \ell_{i2}, \ell_{i3}, \ell_{i4}$ be denoted by $L_i = \{y_{i1}, y_{i2}, y_{i3}, y_{i4}\}$. The sets C_{i1} and C_{i2} are adjacent to y_{i1}. The sets C_{i2} and C_{i3} are adjacent to y_{i2}. The sets C_{i3} and C_{i4} are adjacent to y_{i3}. Additionally, C_{i3} is adjacent to y_{i4}. Further, every vertex in C_i is adjacent to all literal vertices corresponding to literals not in C_i. The union of all clause gadgets C_i is denoted by C and their vertices are called clause vertices.*

- *For all $i \neq j$, the set C_i is adjacent to the set C_j.*

- *For $1 \leq i \leq n$, the vertices in $X_i \setminus \{x_i, \overline{x_i}\}$ are adjacent to $V(G) \setminus X_i$.*

  This completes the construction of the graph G_Φ (see Figure 16 for an example)

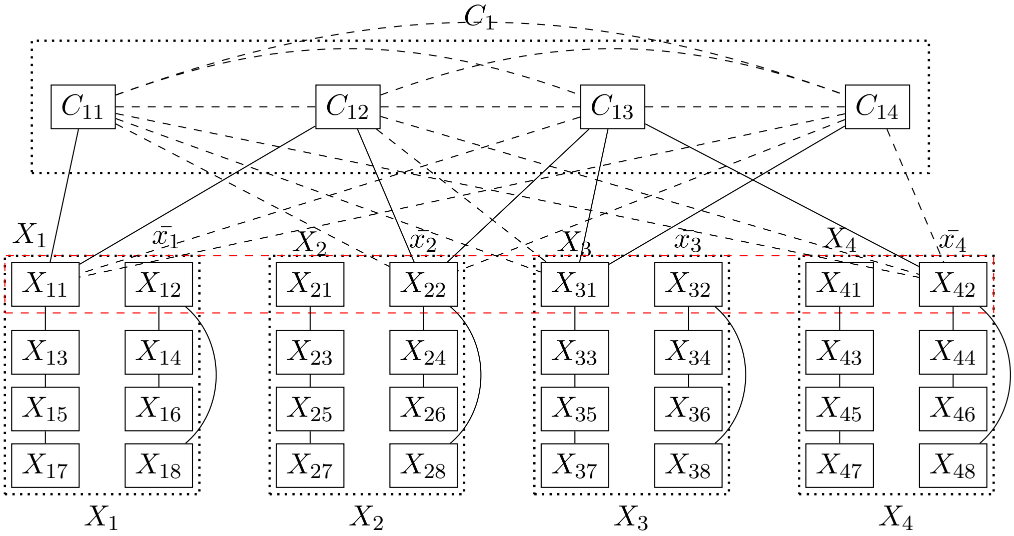


Figure 16: An example of Construction 9 for the formula $\Phi = C_1$, where $C_1 = x_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_4$ corresponding to $C_{1,2,4}$, shown in Figure 15. The bold (respectively dashed) lines connecting two rectangles indicate that each vertex in one rectangle is adjacent (respectively non-adjacent) to all vertices in the other rectangle. If there is no line shown between two rectangles, then the vertices in them are adjacent, with an exception – all the vertices in the red rectangle (dashed) together form an independent set. Similarly, if there is no line shown between two rectangles in the dotted rectangles, then the rectangles in them are non-adjacent.

Lemma 4.32. *Let Φ be a yes-instance of* 4-SAT$_{\geq 2}$ *and ψ be a truth assignment satisfying Φ. Then $G_\Phi \oplus S \in \mathcal{F}(C_{1,2,4})$, where S is the set of literal vertices whose corresponding literals were assigned TRUE by ψ.*

Proof. Let $G_\Phi \oplus S$ contain a $C_{1,2,4}$ induced by A (say). We prove the lemma with the help of a set of claims.

Claim 1: A is not a subset of X_i, for $1 \leq i \leq n$.

Assume that A is a subset of X_i. Since $C_{1,2,4}$ is a prime graph and $\overline{C_{1,2,4}}$ is not isomorphic to $C_{1,2,4}$ as no nontrivial tree is self-complementary, $A \cap Y \leq 1$, where Y is a module isomorphic to $\overline{C_{1,2,4}}$. Thus, $|A \cap X_{ij}|$ is at most one. Therefore, A has nonempty intersection with at least two sets X_{ij} and $X_{i\ell}$. Since $C_{1,2,4}$ has 8 vertices, A has nonempty intersection with each set X_{ij} (for $1 \leq j \leq 8$). Recall that $\{x_i, \overline{x_i}\}$ is not a subset of S. Hence, we obtain that the graph induced by A is $2P_4$, which is a contradiction.

Claim 2: Let $X_i' = X_i \setminus \{x_i, \overline{x_i}\}$ and $\overline{X_i} = V(G_\Phi) \setminus X_i$. If $|A \cap X_i'| \geq 1$, then $A \cap \overline{X_i} = \emptyset$. Similarly, if $|A \cap \overline{X_i}| \geq 1$, then $A \cap X_i' = \emptyset$.

For a contradiction, assume that A contains at least one vertex from X_i' and at least one vertex from $\overline{X_i}$. Since X_i' and $\overline{X_i}$ are adjacent, either $|A \cap X_i'| = 1$ or $|A \cap \overline{X_i}| = 1$. Assume that $A \cap X_i' = \{u\}$. Note that $V(G_\Phi) \setminus (X_i' \cup \overline{X_i}) = \{x_i, \overline{x_i}\}$. Therefore, A contains at least 5 vertices from $\overline{X_i}$. Then the graph induced by A has a $K_{1,5}$, which is a contradiction, as there is no $K_{1,5}$ in $C_{1,2,4}$. Assume that $A \cap \overline{X_i} = \{u\}$. Then with the same argument as given above, we obtain that the graph induced by A contains $K_{1,5}$, which is a contradiction.

Claim 3: A is not a subset of L, the set of all literal vertices.

This follows from the fact that L induces a $K_n + nK_1$ in $G_\Phi \oplus S$.

Claim 4: A cannot have nonempty intersections with three distinct clause gadgets C_i, C_j, and C_ℓ.

Claim 5: There exists no C_i and C_j $(i \neq j)$ such that $|A \cap C_i| \geq 2$ and $|A \cap C_j| \geq 2$.

Claim 4 and 5 follow from the fact that C_i and C_j are adjacent for $i \neq j$ and $C_{1,2,4}$ does have neither a triangle nor a C_4.

Claim 6: A is not a subset of C.

For a contradiction, assume that $A \subseteq C$. By Claim 4, A cannot have nonempty intersections with three distinct clause gadgets C_i, C_j, and C_ℓ. Since $C_{1,2,4}$ is a prime graph and $\overline{C_{1,2,4}}$ is not isomorphic to $C_{1,2,4}$, we obtain that $|A \cap C_{ij}|$ is at most one. Thus, $A \cap C_i$ induces an independent set of size at most four, which implies that A cannot be a subset of C_i. Now assume that A has vertices from exactly two sets C_i and C_j. Since C_i and C_j are adjacent, A induces a star graph which is a contradiction.

Claim 7: If $|A \cap C_i| \geq 2$, then $|A \cap (L \setminus L_i)|$ is at most one.

It follows from the fact that C_i is adjacent to all vertices in $A \cap (L \setminus L_i)$ and $C_{1,2,4}$ does not have a C_4.

We are ready to prove the lemma. By Claim 1, A is not a subset of X_i. By Claim 2, A cannot have vertices from both $X_i \setminus \{x_i, \overline{x_i}\}$ and $\overline{X_i}$. This implies that $A \subseteq L \cup C$. By Claim 3, A cannot be a subset of L and by Claim 6, A cannot be a subset of C. Therefore, A contains vertices from both L and C. By Claim 4, A cannot have nonempty intersections with three distinct clause gadgets C_i, C_j and C_ℓ. Therefore, $A \cap C \subseteq (C_i \cup C_j)$. Assume that A has nonempty intersection with both C_i and C_j. By Claim 5, we can assume that $|A \cap C_j| = 1$ and $|A \cap C_i| \geq 1$.

Let $A \cap C_j = \{c_j\}$. Assume that $|A \cap C_i| = 4$. Then, A induces a graph containing a $K_{1,4}$, which is a contradiction. If $|A \cap C_i| = 3$, then $A \cap C$ induces a $K_{1,3}$, which implies that $A \cap L$ induces a $P_3 + K_1$, which is a contradiction. Let $|A \cap C_i| = 2$. Since $A \cap C$ induces a P_3, $A \cap L$ induces either a $T_{1,2}$, or a $P_4 + K_1$, or a $2K_2 + K_1$, or a $P_3 + P_2$, or a $P_3 + 2K_1$, which is a contradiction. Assume that $|A \cap C_i| = 1$, then $A \cap C$ induces a K_2, which implies that $A \cap L$ induces a graph containing P_a, for $a \geq 3$ which is a contradiction. Thus, it is clear that the vertices in $A \cap C$ are from at most one clause gadget C_i.

If $|A \cap C_i| = 1$, then $A \cap L$ induces a graph containing P_3. Since L induces a $K_n + nK_1$, it leads to a contradiction. If $|A \cap C_i| = 2$, then also $A \cap L$ induces a graph containing a P_3. Let, $|A \cap C_i| = 3$. Then by claim 7, $A \cap (L \setminus L_i)$ is at most one. Let $w \in A \cap (L \setminus L_i)$. Thus, A contains a $K_{1,3}$ with center as w. Therefore, $A \cap L_i$ induces a P_3 which is a contradiction. If $A \cap (L \setminus L_i) = \emptyset$, then $|A \cap L| = 4$. This leads to a contradiction as A contains only seven vertices. Now assume that $|A \cap C_i| = 4$. Then by claim 7, $A \cap (L \setminus L_i)$ is at most one. Let $w \in A \cap (L \setminus L_i)$. Thus, the graph induced by A contains a $K_{1,4}$ with center as w, which is a contradiction. Therefore $A \cap L \subseteq L_i$. Since at least two vertices in $A \cap L_i$ is in S, the graph induced by A contains at least one edge more than that of $C_{1,2,4}$, which gives a contradiction. □

Lemma 4.33. *Let Φ be an instance of* 4-SAT$_{\geq 2}$. *If $G_\Phi \oplus S \in \mathcal{F}(C_{1,2,4})$ for some $S \subseteq V(G_\Phi)$ then there exists a truth assignment satisfying Φ.*

Proof. Let $G_\Phi \oplus S \in \mathcal{F}(C_{1,2,4})$ for some $S \subseteq V(G_\Phi)$. We want to find a satisfying truth assignment of Φ. We know that each of the sets C_{ij}, for $1 \leq i \leq m$ and $1 \leq j \leq 4$, induces a $\overline{C_{1,2,4}}$. Therefore, each such set has at least one vertex not in S. Hence at least two vertices in L_i must belong to S, otherwise there is an induced $C_{1,2,4}$ by vertices in L_i and one vertex each from $C_{ij} \setminus S$, for $1 \leq j \leq 4$.

Similarly, each set X_{ij}, for $1 \leq i \leq n$ and $3 \leq j \leq 8$, induces a $\overline{C_{1,2,4}}$. Therefore, X_{ij} has at least one vertex untouched by S. Hence, if both x_i and $\overline{x_i}$ are in S, then there is an induced copy of $C_{1,2,4}$ in $G_\Phi \oplus S$, which is a contradiction. Therefore, both $\{x_i, \overline{x_i}\}$ is not a subset of S. Now, it is straight-forward to verify that assigning TRUE to each literal corresponding to the literal vertices in S is a satisfying truth assignment for Φ. □

Now, Theorem 4.31 follows from Lemma 4.32 and Lemma 4.33.

We observe that, for any integer $t \geq 4$, the subdivision of claw $C_{1,1,t-2}$ is obtained by introducing a false-twin for a leaf of a P_t. Then, Observation 4.34 follows directly from Lemma 3.1.

Observation 4.34. *There is a linear reduction from* SC-TO-$\mathcal{F}(P_t)$ *to* SC-TO-$\mathcal{F}(C_{1,1,t-2})$.

Next we prove that $\overline{T}$ is 5-connected for all subdivisions of claw T having at least 9 vertices.

Observation 4.35. *Let T be a subdivision of claw. Then $\overline{T}$ is 5-connected if and only if T contains at least 9 vertices.*

Proof. It is trivial to observe that the complement of a forest is disconnected if and only if the forest contains a single tree which is a star graph. Assume that T has at least 9 vertices. Let V' be a subset of vertices such that $\overline{T} - V'$ is disconnected. Then $T - V'$ is a star graph of at most 4 vertices (there are no star graph of 5 vertices induced in T). This implies that $|V'| \geq 5$. Therefore $\overline{T}$ is 5-connected. Now, assume that T has only at most 8 vertices. Then, let V' be the set of vertices not in the unique claw in T. Clearly, $|V'| \leq 4$ and $\overline{T} - V'$ is disconnected. Therefore, $\overline{T}$ is not 5-connected. ☐

Now, we are ready to prove the main result of this section.

Theorem 4.36. *Let $x \leq y \leq z$ be integers such that at least one of the following conditions are satisfied.*

(i) $x = 1, y = 2, z = 4$, *or*

(ii) $x = y = 1$, *and* $z \geq 4$, *or*

(iii) $x + y + z \geq 8$.

Then SC-TO-$\mathcal{F}(C_{x,y,z})$ *is NP-Complete. Further, the problem cannot be solved in time* $2^{o(|V(G)|)}$*, unless the ETH fails.*

Proof. Let T be SC-TO-$\mathcal{F}(C_{x,y,z})$ If $x = 1, y = 2$, and $z = 4$, then the statements follow from Theorem 4.31. If $x = y = 1$, and $z \geq 4$, then by Observation 4.34, there is a linear reduction from SC-TO-$\mathcal{F}(P_{z+2})$ to SC-TO-$\mathcal{F}(C_{x,y,z})$. Then the statements follow from Corollary 4.26. Assume that $y > 1$. Then T is a prime graph. If $x+y+z \geq 8$, then T has at least 9 vertices and by Observation 4.35, $\overline{T}$ is 5-connected. If there is an independent set of size 4 in T, then by Theorem 3.5, SC-TO-$\mathcal{F}(\overline{T})$ is NP-Complete and cannot be solved in subexponential-time, assuming the ETH. Then the statements follow from Proposition 2.4. So, it is sufficient to prove that T has an independent set of size 4. Let c be the unique vertex with degree 3 in T and let $\{c_1, c_2, c_3\}$ be the leaves in T. If $\{c, c_1, c_2, c_3\}$ forms an independent set, then we are done. Otherwise, at least one of the leaves, say c_1 is adjacent to c. Then $T - \{c\}$ contains an isolated vertex c_1 and two nontrivial paths such that one of them has length at least two. Then clearly, there is an independent set of size 4 in $T - c$, and hence in T. ☐

Corollary 4.37 follows directly from the constraints in Theorem 4.36.

Corollary 4.37. *Let T be a subdivision of claw not in $\{C_{1,1,1}, C_{1,1,2}, C_{1,1,3}, C_{1,2,2}, C_{1,2,3}, C_{2,2,2}, C_{2,2,3}\}$. Then* SC-TO-$\mathcal{F}(T)$ *is NP-Complete. Further, the problem cannot be solved in time* $2^{o(|V(G)|)}$*, unless the ETH fails.*

4.7 Putting them together

In this section, we prove the main result (Theorem 4.1) of this paper by using the results proved so far. We need a few more observations.

Observation 4.38. *Let T be a prime tree such that there are two adjacent internal vertices u, v which are not adjacent to any leaf of T. Then either of the following conditions is satisfied.*

(i) T has an independent set of size 4 and $\overline{T}$ is 5-connected, or

(ii) T is either a P_6, or a P_7, or the subdivision of claw $C_{1,2,4}$.

Proof. Let the neighbor of u other than v be u'. Similarly, let the neighbor of v other than u be v'. By the assumption, neither u' nor v' is a leaf. Let $T_{u'}$ be the subtree containing u' in $T - u$, and let $T_{v'}$ be the subtree containing v' in $T - v$. Let x be the number of vertices in $T_{u'}$ excluding u' and let y be the number of vertices in $T_{v'}$ excluding v', i.e., $x = |T_{u'}| - 1$, and $y = |T_{v'}| - 1$. Without loss of generality, assume that $x \leq y$. Since u' and v' are not leaves, we obtain that $x \geq 1$ and $y \geq 1$. If $x = 1$ and $y = 1$, then T is P_6. If $x = 1$ and $y = 2$, then T is P_7. If $x = 1$ and $y = 3$, then T is $C_{1,2,4}$ (recall that T is prime). If $x = 1$ and $y \geq 4$, then there is no subset V' of size at most 4 such that $T - V'$ is a star graph. Therefore, $\overline{T}$ is 5-connected. Further, there is an independent set of size 4 in T. If $x = 2$ and $y = 2$, then T is P_8. Then $\overline{T}$ is 5-connected and T has an independent set of size 4. If $x = 2$ and $y \geq 3$, then T has an independent set of size 4 and $\overline{T}$ is 5-connected. The case is same when $x \geq 3$. ☐

Now, we are ready to prove the main theorem.

Proof of Theorem 4.1. Let T' be the internal tree and Q_T be the quotient tree of T. Let p be the number of internal vertices of T. If $p = 1$, then T is a star graph and the statements follow from Proposition 2.3. If $p = 2$, then T is a bistar graph and the statements follow from Theorem 4.13. If $p = 3$, then T is a tristar graph and the statements follow from Theorem 4.17. Assume that $p \geq 4$. If T has only two leaves, then T is isomorphic to P_ℓ, for $\ell \geq 6$. Then the statements follow from Corollary 4.26. If T has exactly three leaves, then T is a subdivision of claw. Then the statements follow from Corollary 4.37. Assume that T has at least four leaves.

If Q_T has two adjacent internal vertices which are not adjacent to any leaves, then by Observation 4.38, either (i) Q_T has an independent set of size 4 and $\overline{Q_T}$ is 5-connected or (ii) Q_T is either a P_6, or a P_7, or a $C_{1,2,4}$. If (i) is true, then SC-TO-$\mathcal{F}(\overline{Q_T})$ is NP-Complete and cannot be solved in subexponential-time (assuming the ETH). Then, so is for SC-TO-$\mathcal{F}(Q_T)$, by Proposition 2.4. Then the statements follow from Lemma 3.1. If (ii) is true, then the statements follow from Corollary 4.26, Theorem 4.31, and Lemma 3.1. Therefore, assume that Q_T has no two adjacent internal vertices not adjacent to any leaves of Q_T. Hence, T has no two adjacent internal vertices not adjacent to any leaves of T. Then, if T' (the internal tree) is not a star graph, then the statements follow from Theorem 4.5. Assume that T' is a star graph. Assume that the condition (i) of Theorem 4.5 is not satisfied, i.e., the center of T' has at least one leaf of T as neighbor, one leaf of T' has exactly one leaf of T as neighbor, and T is neither $C_{1,2,2,2}$ nor $C_{1,2,2,2,2}$. Assume that T has exactly 4 internal vertices. Then T' is a claw and Q_T is $C_{1,2,2,2}$. Then by Theorem 4.5, SC-TO-$\mathcal{F}(Q_T)$ is NP-Complete and cannot be solved in subexponential-time (assuming the ETH). Then the statements follow from Lemma 3.1. Similarly, when T has exactly 5 internal vertices, we obtain that Q_T is $C_{1,2,2,2,2}$ and then the statements follow from Theorem 4.5 and Lemma 3.1. Assume that T has at least 6 internal vertices. Then, T' is a $K_{1,a}$, for some $a \geq 5$. Then by Lemma 4.2, there is a linear reduction from SC-TO-$\mathcal{F}(K_{1,a})$ to SC-TO-$\mathcal{F}(Q_T)$. By Proposition 2.3, SC-TO-$\mathcal{F}(K_{1,a})$ is NP-Complete and cannot be solved in subexponential-time (assuming the ETH). Then the statements follow from Lemma 3.1. ☐

5 Polynomial-time algorithm

In this section, we prove that SC-TO-$\mathcal{F}(paw)$ can be solved in polynomial-time. We use a result by Olariu [12] that every component of a paw-free graph is either triangle-free or complete mutitpartite.

Proposition 5.1 ([12]). *A graph G is a paw-free if and only if each component of G is either triangle-free or complete multipartite.*

A graph is complete multipartite if and only if it does not contain any $K_2 + K_1$. It is known that SC-TO-$\mathcal{F}(K_3)$ and SC-TO-$\mathcal{F}(K_2 + K_1)$ can be solved in polynomial-time. The former is proved in [7] and the latter is implied by another result from [7] that Subgraph Complementation problems admit polynomial-time algorithms if the target graph class is expressible in MSO_1 and has bounded clique-width.

Proposition 5.2 ([7]). SC-TO-$\mathcal{F}(K_3)$ *and* SC-TO-$\mathcal{F}(K_2 + K_1)$ *are solvable in polynomial-time.*

Our algorithm works as follows: If there is a solution which transforms the input graph into a single paw-free component, then we use Proposition 5.2. If S transforms G into multiple components, then we guess vertices belonging to a constant number of those components and then try to obtain S by analysing the neighborhood of the guessed vertices. We also use the following two observations, which say that it is safe to assume that the input graph does not contain any independent module or clique module of size at least 4.

Observation 5.3. *Let G be a graph having an independent module I of size at least 4. Let G' be the graph obtained from G by removing I and introducing an independent module I' of size 3 with the same adjacency as that of I. Then, G is a yes-instance of* SC-TO-$\mathcal{F}(paw)$ *if and only if G' is a yes-instance of* SC-TO-$\mathcal{F}(paw)$.

Proof. Let G be a yes-instance. Let S be a solution of G. Initialize $S' = S \setminus I$. Include in S' vertices from I' in such a way that $|S' \cap I'| = |S \cap I|$, if $0 \leq |S \cap I| \leq 1$, and $|S' \cap I'| = 2$, if $2 \leq |S \cap I| \leq |I| - 1$, and $|S' \cap I'| = 3$, if $I \subseteq S$. We claim that S' is a solution of G'. For a contradiction, assume that there is a set $A' \subseteq V(G')$ which induces a paw in $G' \oplus S'$. If A' has no vertices from I', then A' induces a paw in $G \oplus S$, which is a contradiction. Therefore, $A' \cap I' \neq \emptyset$. Initialize A to be $A' \setminus I'$. Assume that $|A' \cap I'| = 1$. Let $A' \cap I' = \{u'\}$. Then, include in A a vertex from $I \cap S$ if $u' \in S'$, and a vertex from $I \setminus S$, if $u' \notin S'$. Then A induces a paw in $G \oplus S$. Let $|A' \cap I'| = 2$. Since there is no independent module of size 2 in a paw, we obtain that at least one vertex of $A' \cap I'$ must be in S'. If both vertices in $A' \cap I'$ is in S', then include two vertices of $S \cap I$ in A. Then A induces a paw in $G \oplus S$. If one vertex of $A' \cap I'$ is in S' and the other is not in S', then include one vertex from $S \cap I$ and one vertex from $I \setminus S$ in A. Then A induces a paw in $G \oplus S$. Let $|A' \cap I'| = 3$. Since paw does not have an independent module of size 2, we obtain that at least two vertices of $A' \cap I'$ are in S'. Since the triangle in a paw is not a module, we obtain that not all the three vertices in $A' \cap I'$ can be in S'. Therefore, exactly two vertices in $A' \cap I'$ are in S'. Then, include two vertices of $S \cap I$ and one vertex of $I \setminus S$ in A. Then A induces a paw in $G \oplus S$.

For the other direction, assume that S' is a solution of G'. Initialize S to be $S' \setminus I'$. Include vertices from I to S in such a way that $|S \cap I| = |S' \cap I'|$, if $0 \leq |S' \cap I'| \leq 2$, $|S \cap I| = |I|$ if $I' \subseteq S'$. We claim that S is a solution for G. For a contradiction, assume that $A \subseteq V(G)$ induces a paw in $G \oplus S$. If A has no vertices from I, then A induces a paw in $G' \oplus S'$, which is a contradiction. Therefore, $A \cap I \neq \emptyset$. Initialize A' to be $A \setminus I$. Let $|A \cap I| = 1$. Let $A \cap I = \{u\}$. If $u \in S$, then include a vertex $u' \in S' \cap I'$ in A'. If $u \notin S$, then include a vertex $u' \in I' \setminus S'$ in A'. Then, A' induces a paw in $G' \oplus S'$. Let $|A \cap I| = 2$. Since a paw does not have an independent module of size 2, we obtain that at least one vertex of $A \cap I$ is in S. Assume that exactly one vertex of $A \cap I$ is in S. Then, include in A', one vertex from $I' \cap S'$ and one vertex from $I' \setminus S'$. Then A' induces a paw in $G' \oplus S'$. If both the vertices in $A \cap I$ are in S, then include in A' two vertices in $I' \cap S'$. Then A' induces a paw in $G' \oplus S'$. Assume that $|A \cap I| = 3$. As observed before, at least two vertices in $A \cap I$ must be in S. Since the triangle in a paw is not a module, not all three vertices in $A \cap I$ can be in S. Therefore, exactly two vertices of $A \cap I$ is in S. Then, include in A' two vertices from $I' \cap S'$ and one vertex from $I' \setminus S'$. Then, A' induces a paw in $G' \oplus S'$. We observe that all the 4 vertices in A cannot be from I. This completes the proof. ☐

The proof of Observation 5.4 is similar.

Observation 5.4. *Let G be a graph having a clique module I of size at least 4. Let G' be the graph obtained from G by removing I and introducing a clique module I' of size 3 with the same adjacency as that of I. Then, G is a yes-instance of* SC-TO-$\mathcal{F}$(*paw*) *if and only if G' is a yes-instance of* SC-TO-$\mathcal{F}$(*paw*).

Proof. Let G be a yes-instance. Let S be a solution of G. Initialize $S' = S \setminus I$. Include in S' vertices from I' in such a way that $|S' \cap I'| = |S \cap I|$, if $0 \leq |S \cap I| \leq 1$, and $|S' \cap I'| = 2$, if $2 \leq |S \cap I| \leq |I| - 1$, and $|S' \cap I'| = 3$, if $I \subseteq S$. We claim that S' is a solution of G'. For a contradiction, assume that there is a set $A' \subseteq V(G')$ which induces a paw in $G' \oplus S'$. If A' has no vertices from I', then A' induces a paw in $G \oplus S$, which is a contradiction. Therefore, $A' \cap I' \neq \emptyset$. Initialize A to be $A' \setminus I'$. Assume that $|A' \cap I'| = 1$. Let $A' \cap I' = \{u'\}$. Then, include in A a vertex from $I \cap S$ if $u' \in S'$, and a vertex from $I \setminus S$, if $u' \notin S'$. Then A induces a paw in $G \oplus S$. Let $|A' \cap I'| = 2$. Since there is no independent module of size 2 in a paw, we obtain that at least one vertex of $A' \cap I'$ is not in S'. If neither of the two vertices in $A' \cap I'$ is in S', then include two vertices of $I \setminus S$ in A. Then A induces a paw in $G \oplus S$. If one vertex of $A' \cap I'$ is in S' and the other is not in S', then include one vertex from $S \cap I$ and one vertex from $I \setminus S$ in A. Then A induces a paw in $G \oplus S$. Let $|A' \cap I'| = 3$. Since paw does not have an independent module of size 2, we obtain that at least two vertices of $A' \cap I'$ are not in S'. Since the triangle in a paw is not a module, we obtain that at least one vertex in $A' \cap I'$ must be in S'. Therefore, exactly two vertices in $A' \cap I'$ are not in S'. Then, include two vertices of $I \setminus S$ and one vertex of $I \cap S$ in A. Then A induces a paw in $G \oplus S$.

For the other direction, assume that S' is a solution of G'. Initialize S to be $S' \setminus I'$. Include vertices from I to S in such a way that $|S \cap I| = |S' \cap I'|$, if $0 \leq |S' \cap I'| \leq 2$, and $|S \cap I| = |I|$ if $I' \subseteq S'$. We claim that S is a solution for G. For a contradiction, assume that $A \subseteq V(G)$ induces a paw in $G \oplus S$. If A has no vertices from I, then A induces a paw in $G' \oplus S'$, which is a contradiction. Therefore, $A \cap I \neq \emptyset$. Initialize A' to be $A \setminus I$. Let $|A \cap I| = 1$. Let $A \cap I = \{u\}$. If $u \in S$, then include a vertex $u' \in S' \cap I'$ in A'. If $u \notin S$, then include a vertex $u' \in I' \setminus S'$ in A'. Then, A' induces a paw in $G' \oplus S'$. Let $|A \cap I| = 2$. Since a paw does not have an independent module of size 2, we obtain that at least one vertex of $A \cap I$ is not in S. Assume that exactly one vertex of $A \cap I$ is in S. Then, include in A', one vertex from $I' \cap S'$ and one vertex from $I' \setminus S'$. Then A' induces a paw in $G' \oplus S'$. If neither of the two vertices in $A \cap I$ is in S, then include in A' two vertices in $I' \setminus S'$. Then A' induces a paw in $G' \oplus S'$. Assume that $|A \cap I| = 3$. As observed before, at least two vertices in $A \cap I$ must not be in S. Since the triangle in a paw is not a module, at least one vertex in $A \cap I$ is in S. Therefore, exactly two vertices of $A \cap I$ are not in S. Then, include in A' one vertex from $I' \cap S'$ and two vertices from $I' \setminus S'$. Then, A' induces a paw in $G' \oplus S'$. We observe that all the 4 vertices in A cannot be from I. This completes the proof. ☐

It is trivial to note that removing paw-free components from the input graph is safe.

Proposition 5.5. *Let G be a graph such that G' is a connected paw-free component of it. Then G is a yes-instance of* SC-TO-$\mathcal{F}$*(paw) if and only if $G - V(G')$ is a yes-instance of* SC-TO-$\mathcal{F}$*(paw).*

Now onward, we assume that G has no paw-free component and no independent or clique module of size at least 4. Step 1 of our algorithm takes care of the case when G is paw-free. If G is not paw-free, then every solution S will have at least two vertices. Step 2 takes care of the case when there is a solution which transforms the input graph into a single paw-free component. Step 3 handles the case when there are at least three components in the resultant graph. Step 4 resolves the case when there are exactly two components.

For integers $p, q \geq 1$, a (p, q)-split partition of a graph G is a partition of its vertices into two sets P, Q such that the maximum clique size of the subgraph induced by P is at most p and the maximum independenet set of the subgraph induced by Q is at most q. If a graph admits a (p, q)-split partition, then the graph is known as a (p, q)-split graph. It is known that all (p, q)-split graphs can be recognized and all (p, q)-split partitions of them can be found in polynomial-time (see [8, 13, 14]). We define a component partition of a graph G as a partition of its vertices into two sets P, Q such that P induces a single component or an independent set of size at most 3, an Q contains the remaining vertices. We observe that all component partitions of a graph can be found in polynomial-time.

Algorithm for SC-TO-$\mathcal{F}$(*paw*)
Input: A graph G.
Output: If G is a yes-instance of SC-TO-$\mathcal{F}$(*paw*), then returns YES;
returns NO otherwise.

Step 1 : If G is paw-free, then return YES.

Step 2 : If G is a yes instance of SC-TO-$\mathcal{F}(K_3)$, or a yes-instance of SC-TO-$\mathcal{F}(K_2 + K_1)$, then return YES.

Step 3 : For every triangle u, v, w in G, if $(N(u) \cap N(v)) \cup (N(u) \cap N(w)) \cup (N(v) \cap N(w))$ is a solution, then return YES.

Step 4 : For every ordered pair of adjacent vertices (u, v), do the following:

(i) Compute L_u and L_v, the lists of $(1, 2)$-split partitions of $N(u)$ and $N(v)$ respectively.
(ii) Compute R_u and R_v, the lists of component partitions of $N(u)$ and $N(v)$ respectively.
(iii) For every (P_u, Q_u) in L_u, and for every (P_v, Q_v) in L_v, if $Q_u \cup Q_v$ is a solution, then return YES.
(iv) For every (X_u, Y_u) in R_u, and for every (X_v, Y_v) in R_v, if $Y_u \cup Y_v$ is a solution, then return YES.

(v) For every (X_v, Y_v) in R_v, do the following:

(a) For every set $Y'_v \subseteq Y_v$ of size at least $|Y_v| - 2$, do the following:

i. Initialize $Z = \{v\}$, and let I be the isolated vertices in $G[N(u)]$.

ii. Add $N(u) \setminus I$ to Z.

iii. Add to Z every vertex in I adjacent to some vertex in X_v.

iv. If $Y'_v \cup Z$ is a solution, then return YES.

Step 5 : Return NO.

Next few lemmas state that the algorithm returns YES in various cases.

Lemma 5.6. *If there exists a set $S \subseteq V(G)$ such that $G \oplus S$ is a connected paw-free graph, then the algorithm returns YES.*

Proof. By Proposition 5.1, a connected paw-free graph is either triangle-free or complete multipartite. Hence $G \oplus S$ is triangle-free or complete multipartite. Recall that the complete multipartite graphs are exactly the class of $K_2 + K_1$-free graphs. Then the algorithm returns YES at Step 2. ☐

Let S be any solution of G. Let $G_1, G_2, \ldots, G_t$, for some integer $t \geq 1$, be the connected components of $G \oplus S$. Let S_i be the intersection of S with G_i (for $1 \leq i \leq t$). For a vertex $u \in S_i$, by A_u we denote the neighbors of u in $G_i - S_i$. Note that S_i and S_j are adjacent in G for $i \neq j$.

Lemma 5.7. *Assume that there exists a set $S \subseteq V(G)$ such that $G \oplus S$ is a disjoint union of at least three connected components. Then the algorithm returns YES.*

Proof. Let S be a set as specified in the lemma. Let $u \in S_1, v \in S_2$, and $w \in S_3$. Note that $N(u) \cap N(v) \subseteq S$ and $S \setminus (S_1 \cup S_2) \subseteq N(u) \cap N(v)$. Therefore, $(N(u) \cap N(v)) \cup (N(u) \cap N(w)) \cup (N(v) \cap N(w)) = S$. Then the algorithm returns YES at Step 3. ☐

Lemma 5.8. *Assume that there exists a set $S \subseteq V(G)$ such that $G \oplus S$ is paw-free and $G \oplus S$ has exactly two components and both of them are complete multipartite. Then the algorithm returns YES.*

Proof. Let S be a solution as specified in the lemma. By Propositon 5.5, we can safely assume $S_1, S_2 \neq \emptyset$. Let $u \in S_1$ and $v \in S_2$. Both $G_1 - S_1$ and $G_2 - S_2$ are complete multipartite graphs. Let $I_1, I_2, \ldots I_{t_1}$ be the independent set partition of $G_1 - S_1$, and let $J_1, J_2, \ldots, J_{t_2}$ be the independent set partition of $G_2 - S_2$. Clearly, each set I_a (for $1 \leq a \leq t_1$) and J_a (for $1 \leq a \leq t_2$) are independent modules in G. Therefore, each of them has size at most 3. Therefore, if A_u has at least 4 vertices, then A_u induces a connected component in G. Hence, if A_u is disconnected, then it forms an independent set of size at most 3. Similarly, if A_v has at least 4 vertices, then A_v induces a connected component in G, and if A_v induces a disconnected graph, then it forms an independent set of size at most 3. Therefore, $(A_u, N(u) \setminus A_u)$ is a component partition of $N(u)$. Similarly, $(A_v, N(v) \setminus A_v)$ is a component partition of $N(v)$. Note that $S = (N(u) \setminus A_u) \cup (N(v) \setminus A_v)$. Therefore, at Step 4(iv), the algorithm returns YES. ☐

Lemma 5.9. *Assume that there exists a set $S \subseteq V(G)$ such that $G \oplus S$ is paw-free and $G \oplus S$ has exactly two components and both are triangle-free. Then the algorithm returns YES.*

Proof. Let S be a solution as specified in the lemma. By Proposition 5.5, we can safely assume $S_1, S_2 \neq \emptyset$. Let $u \in S_1$ and $v \in S_2$. Both $G_1 - S_1$ and $G_2 - S_2$ are triangle-free graphs. Clearly, A_u is an independent set (i.e., K_2-free)- otherwise there is a triangle in $G_1 \oplus S_1$ formed by u and two of its adjacent neighbors in A_u. Further, S induces $3K_1$-free graphs in G, as both S_1 and S_2 are $3K_1$-free and they are adjacent. Therefore, $(A_u, N(u) \setminus A_u)$ is a $(1,2)$-split partition of $N(u)$, and $(A_v, N(v) \setminus A_v)$ is a $(1,2)$-split partition of $N(v)$. Note that, $(N(u) \setminus A_u) \cup (N(v) \setminus A_v) = S$. Therefore, the algorithm returns YES at Step 4(iii). ☐

Lemma 5.10. *Assume that there exists a set $S \subseteq V(G)$ such that $G \oplus S$ is paw-free and $G \oplus S$ has exactly two components G_1 and G_2 such that G_1 is triangle-free and G_2 is complete multipartite. Then the algorithm returns YES.*

Proof. Let $u \in S_1$ and $v \in S_2$. Note that $(A_v, N(v) \setminus A_v)$ is a component partition of $N(v)$ (A_v induces either a connected graph or has at most three vertices (which forms an independent set)). Therefore, in some iteration of the loop at Step 4(v), we get $X_v = A_v$ and $Y_v = S_1 \cup (N(v) \cap S_2)$. By Observation 5.4, there cannot be any clique module of size at least 4. Note that $(N(v) \cup \{v\}) \cap S_2$ is a clique module. Therefore, only at most two vertices of $N(v) \setminus A_v$ are not in S_1. Further, $S_1 \subseteq N(v) \setminus A_v$. Therefore, in some iteration of the loop at Step 4(v)(a), we get $Y'_v = S_1$. The vertex u cannot have two adjacent neighbors in $G_1 - S_1$ (otherwise, there will be a triangle in $G_1 \oplus S_1$). Therefore, every vertex $a \in N(u)$ which is having a common neighbor with u must be in S_2. Similarly, every neighbor of u adjacent to a vertex in X_v is in S_2. By Step 4(v)(a)iii of the algorithm, we obtain all vertices of Z except possibly the set I' of isolated vertices in $G[N(u)]$ which are not adjacent to any vertex in X_v. It can be verified that $S \setminus I'$ is also a solution. Therefore, the algorithm return YES at Step 4(v)(a)iv. ☐

Lemma 5.11. *The algorithm returns YES if and only if G is a yes-instance.*

Proof. Clearly, if the algorithm returns YES, then G is a yes-instance. For the other direction, assume that G is a yes-instance. If there exists a solution S such that $G \oplus S$ is a connected component, then the algorithm returns YES by Lemma 5.6. If there exists a solution S such that $G \oplus S$ has at least three connected components, then by Lemma 5.7, the algorithm returns YES. Assume that there is no solution that transforms the graph into a single component or at least three components. Then there must exists a solution S such that $G \oplus S$ has exactly two connected components G_1 and G_2. By Proposition 5.1, each of G_1 and G_2 is either triangle-free or complete multipartite. Assume that both G_1 and G_2 are complete multipartite. Then by Lemma 5.8, the algorithm returns YES. Assume that both G_1 and G_2 are triangle-free. Then by Lemma 5.9, the algorithm returns YES. Now, assume that G_1 is triangle-free and G_2 is complete multipartite. Then the algorithm returns YES by Lemma 5.10. This completes the proof. ☐

Now, Theorem 5.12 follows from Lemma 5.11 and the fact that all split partitions and all component partitions of a graph can be obtained in polynomial-time.

Theorem 5.12. SC-TO-$\mathcal{F}$(*paw*) *can be solved in polynomial-time.*

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